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DATA ASSOCIATION

IN AN OCEAN ENVIRONMENT (U)

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BY

V.T. GABRIEL, R.D. BERLIN AND Y. BAR-SHALOM

SEPTEMBER 1981

PREPARED UNDER CONTRACT N00014-79-C-0528, TASK NR 274-314

FOR

NAVAL ANALYSIS PROGRAM
OFFICE OF NAVAL RESEARCH

GENERAL ELECTRIC COMPANY
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A multisensor data association technique was developed utilizing an expected likelihood approach to link time discontinuous tracks or those simultaneously held by multiple sensors. Nonlinear propagation loss/detection performance, prior target speed/heading distributions, gaussian measurement errors and frequency spectrum comparison are included in the resulting algorithm. Single/multiple lookback techniques are used for link decisions. The technique is considered practical for the low data rates typical of ASW.

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SECTION 1

INTRODUCTION AND SUMMARY

1.1 PURPOSE

The purpose of this study is to develop a multi-sensor, multi-target data association technique applicable to ASW platforms utilizing long range passive/active sonars in a convergence zone ocean environment.

1.2 GENERAL PROBLEM DESCRIPTION

Advanced passive sonar systems can detect and track targets at long range while operating in a convergence zone (CZ) multi-target environment. These tracks are not continuous, hence inter/intra CZ fading gives rise to the need for Temporal Association. Since these tracks may be viewed simultaneously by different sensors, Mutual Association is also required.

For many operations, only passive sonars are used; however, provision must be made for active sonar, surface radar and sensors on other/supporting platforms. With passive operation, bearing and frequency tracks are assumed to be smoothed to obtain at least the mean bearing/frequency and corresponding rates.

Two target types are common; submarines and surface ships. Typical, North Atlantic, surface shipping tracks of 12-hour duration are shown in Figure 1-1. A typical bearing-time history for a long range passive sonar in the circled area is shown in Figure 1-2. The gaps/fades in the tracks are due to the high propagation loss that exists between convergence zones. The long-term data association problem is to bridge these gaps using the bearing-time data shown as well as other available measurables such as bearing rate, frequency rate, active sonar range/bearing and surface (radar) data.

1.3 BACKGROUND

In 1977, a GE study sponsored by NOSC[1] addressed the data association problem for a surface ship employing hull and towed arrays. The development of the Expected Likelihood approach started at that time. A two-stage look back approach was simulated using the data of Figure 1-2. Although many assumptions/simplifications were made, the approach appeared to be better matched to the problem than other known/existing techniques. Essentially all of the existing techniques assume good observability and Maximum Likelihood related measures. SI Chou discusses the Expected Likelihood approach in a recent NOSC report[3].

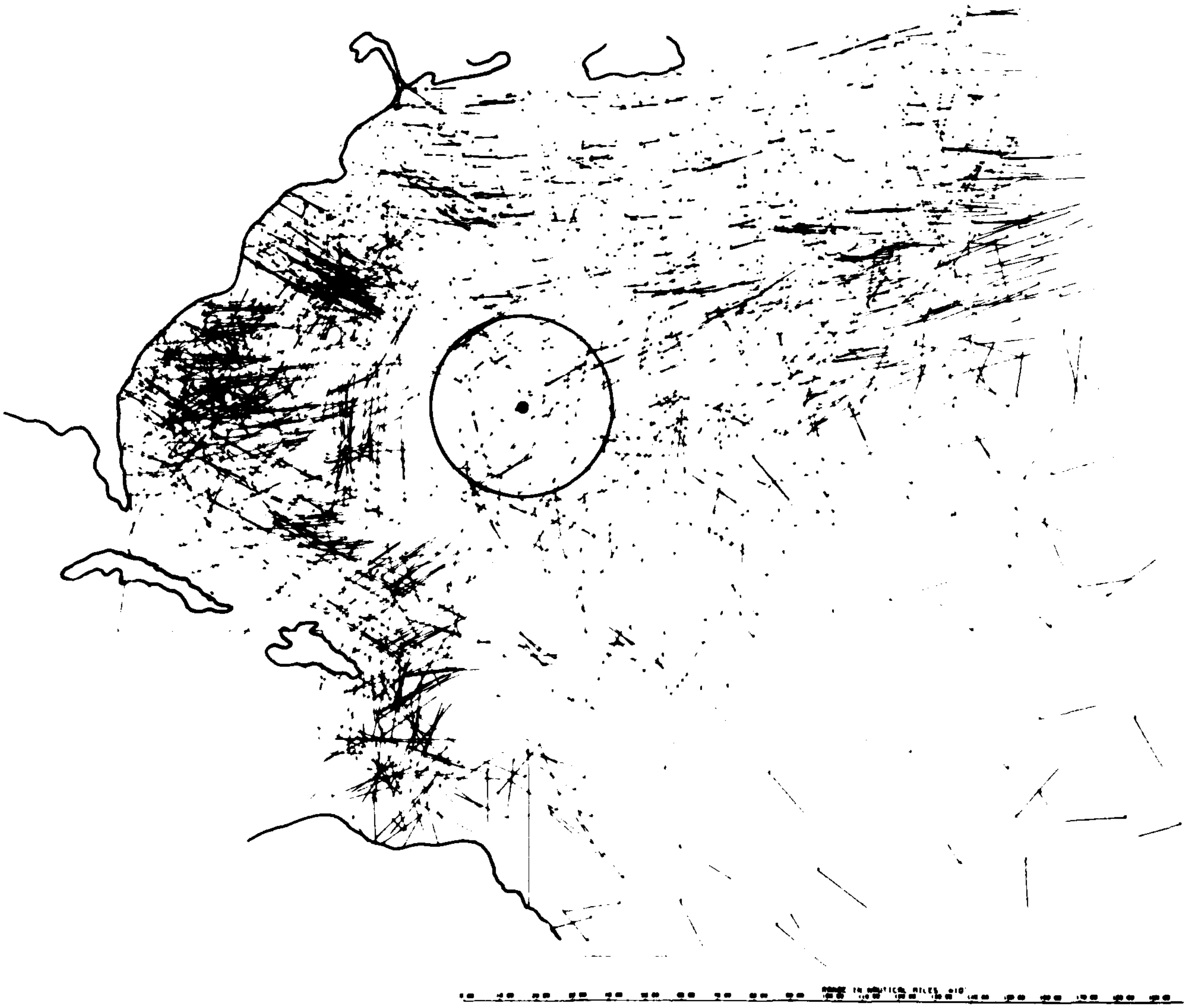

Figure 1-1. Shipping Environment

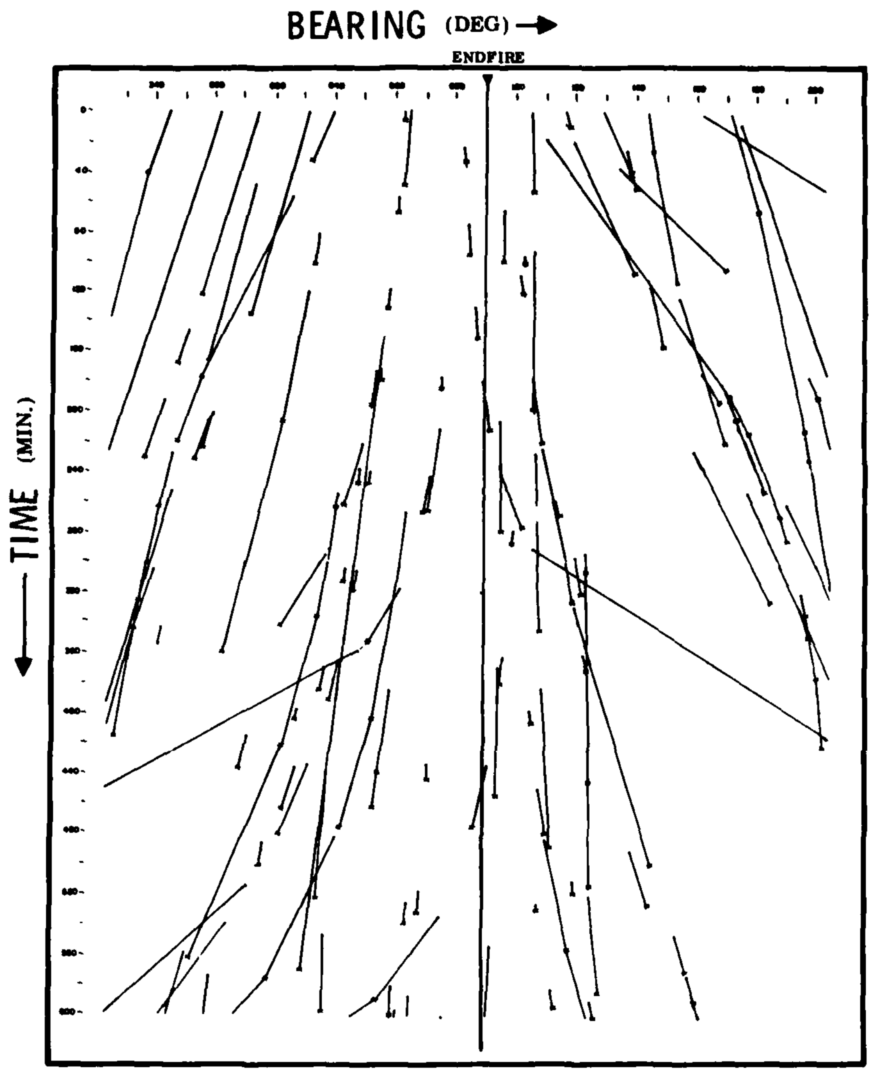

Figure 1-2. Bearing Time History

Concurrently with the above study[1], a data association process was developed in the GE simulation facility for surface ship ASW. It was based on a weighted sum of geometric, frequency and classification correlation indices (not based on likelihood techniques). Recently some of the bearing, frequency techniques described in this report were compared with this process and found to be an improvement[2].

1.4 SUMMARY

Starting with probability of association as the fundamental association measure (utilizing prior target and environment models), the Expected Likelihood Ratio emerges as the key measure. This measure is the weighted average of the likelihood function with averaging taking place over elements of the target state vector; the weights are the prior probability density function of these elements.

Complexity was reduced by careful choice of the state variables and appropriate simplifications. For long term (CZ to CZ) association, the best variable choice was considered to be range/bearing at two time points (reference time (t_o) and candidate time (t_j)). The averaging process then reduced to an integration over only two variables; the range pair ($R_o R_j$). For short term or simultaneous/mutual association the preferred state variable set was range, bearing, target speed and heading. The reduction again resulted in a two variable integration process; range and heading.

Thus, two Expected Likelihood Ratio algorithms evolved: (1) long term, applicable when Δt is long (e.g., greater than 30 minutes) and (2) short term (e.g., Δt less than 30 minutes).

The long term (CZ - CZ) algorithm is given in Section 4. In the numerator note that array gain/propagation loss is factored into the detection term, target speed/heading priors in the target term, geometric measurements in the F_y term and frequency spectra in the last term. Integration is over the range pair ($R_o R_j$) where R_o is range at the time of the reference segment (o) and R_j is range at the time of the jth candidate. The denominator is a relatively simple normalizer with one integral for each segment. The short term algorithm is similar in form, but differs in detail as described in Appendix A6.

1.5 ASSUMPTIONS

a) All targets are of the same general class and hence no distinction is made between surface ship and submarine targets.

b) Targets are non-maneuvering.

c) Measurement errors are gaussian and independent.

d) A priori target/environment statistics are known.

e) Geometric and frequency spectrum data are the only measurables/attributes; classification, demon data, etc., are not included.

f) Segment estimates of mean bearing, bearing rate, etc., used as inputs to the association process have previously been correctly correlated/associated and bad data edited out.

1.6 CONCLUSIONS AND RECOMMENDATIONS

The Expected Likelihood technique developed in this study provides a practical approach to the problem of associating data segments with diverse attributes; geometric and frequency spectrum were included but others can be added. Prior target statistics (e.g., speed/heading) and detection/environmental prediction (e.g., probability of detection versus range) are incorporated with few restrictions on the form of the statistical models. The formulation does not depend on having a range estimate and other initial conditions and hence will work long before localization solutions become available. The formulation is general enough to include the mutual and temporal association of data segments from different platforms and hence applicable to a wide variety of sonar/radar sensors including single/multiple ship hull/towed arrays, offboard sonobouy/arrays as well as radar and other surface sensors.

Recommendations for further study are as follows:

a) For a given current segment a likelihood ratio is computed for each past segment, one at a time; a so called single stage process. A significant improvement is expected when the two stage (two past segments) process is implemented.

b) Additional attributes should be included in the likelihood ratio formulation including at least: Demon signatures and classification decisions. The current study assumed that all targets are of a given class, namely surface ships at constant course and speed; maneuvering targets/subs should also be included in the process.

c) Another important improvement is to determine how to make better association decisions given the likelihood ratio measures. Some work was done in the current study but more is requ. .d.

d) Trade studies should be conducted so that design decisions can be made. A good example of this is: should the amplitude or clipped frequency spectrum be implemented? This is a performance vs complexity trade.

e) The ASW simulator should be further exercised to provide evaluations of the association techniques for trade studies, technique tuning and assessing the payoffs for improvements.

f) A study should be performed to determine the information/data rate currently available from existing sensors/platforms/data links. The payoff for more data and/or additional sensors should be determined to provide planning/requirements inputs.

SECTION 2

FORMULATION OF ASSOCIATION MEASURES

As a target is tracked through a convergence zone, the resulting smoothed data set is defined as a segment. For a typical ith segment at time t_i, the data is defined by Z_i or $Z_i(t_i)$. Three types of data are included: Detection (D_i), Geometric (Y_i) and Frequency Spectrum (S_i) so that

$$Z_i = [D_i \; Y_i \; S_i] \tag{1.1}$$

A current or reference segment is identified by subscript o (i.e., Z_o), a candidate segment set by subscript j (i.e., Z_j) and the remaining segments by subscript k (i.e., Z_k). Thus, the total data set is defined by the alternative forms.

$$Z = Z_{ojk} = [Z_{oj} \; Z_k] = [Z_o \; Z_j \; Z_k]$$

In general, the candidate segment set can include more than one segment. These definitions are illustrated in Figure 2-1.

The problem is to find the best match between a current/reference segment and various candidate segment sets (j = 1,2,...).

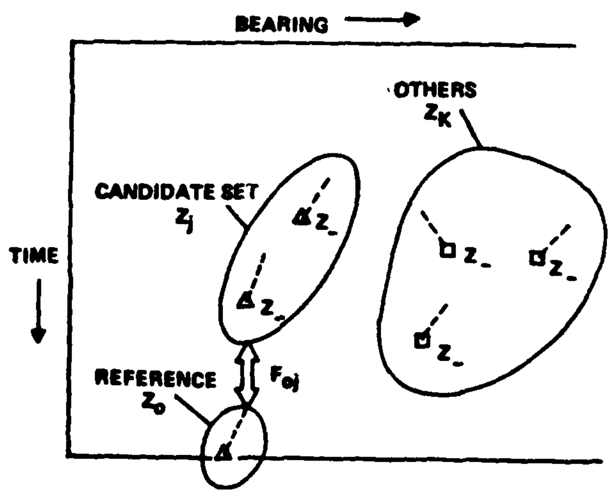

Figure 2-1. Segments

Consider the general association problem of determining "how well" a current data segment (Z_o) matches with the jth candidate (Z_j). The measure of association is given by

$$F_{oj} = P(A_{oj} \mid Z) \tag{2.1}$$

This is the probability of association of the o and jth segment set conditioned on observing the complete segment set (Z).

As shown in A1[1], application of Bayes' rule and manipulation yields

$$\boxed{F_{oj} = \frac{\Lambda_{oj} P(A_{oj})}{P(Z) / P(Z \mid \bar{A})}} \tag{2.2}$$

The key element is the Expected Likelihood Ratio (Λ_{oj}) and hence the focus of much of the subsequent development. The other numerator term $P(A_{oj})$ is the a priori probability of association of the current/candidate set (Z_{oj}). The denominator is a function of the total data set (Z) and hence is a normalizing constant during subsequent comparison/thresholding of the candidate sets. P(Z) is the unconditional density function of Z. $\bar{A}$ is the condition that none of the segments are associated.

If the formulation allows for only a single past segment then, as developed in A1

$$F_{oj} = \frac{\Lambda_{oj}}{\left(\frac{1-P_g}{P_g}\right)m + \sum_{i}^{m} \Lambda_{oi}} \tag{2.3}$$

where P_g is the probability that a past detection occurs and m is the number of past segments observed.

(1) Throughout this report A1, A2, B1, etc. are appendices. For instance, A2 is Appendix A, paragraph 2. When an appendix equation is referenced, it is designated by ( ), i.e., A2 (4).

LIKELIHOOD RATIO

The expected likelihood ratio for the current o and jth segment set is given by (refer to A1)

$$\Lambda_{oj} = \frac{P(Z_{oj} \mid A_{oj})}{P(Z_{oj} \mid \bar{A}_{oj})} \tag{2.4}$$

To simplify notation the oj subscripts will be omitted and n, d used to designate the numerator denominator so that

$$\Lambda = \Lambda_n / \Lambda_d \tag{2.4a}$$

This expresses the a posteriori probability of receiving Z_{oj} with the condition that the segments of Z_{oj} have a common origin and hence are associated (A_{oj}). The non association condition ($\bar{A}_{oj}$) applies to the denominator.

A state vector (X) is used to predict the measurables (Z).

The approach used to develop the numerator and denominator is to write the joint measurement/state probability density function (pdf) and then integrate out the state variables, thus as shown in A2,

$$\boxed{\Lambda = \frac{\int_X P(Z_{oj} \mid X)\, P(X)\, dX}{\left[\int_{X_o} P(Z_o \mid X_o)\, P(X_o)\, dX_o\right] \cdot \left[\int_{X_j} P(Z_j \mid X_j)\, P(X_j)\, dX_j\right]}} \tag{2.5}$$

where the association condition is implied for all P( ).

The conditional density functions (i.e., $P(Z_o \mid X_o)$) are likelihood functions and the others (i.e., $P(X_o)$) are the a priori (prior) probability density functions of the state vectors (i.e., X_o).

$P(Z_{oj} \mid X)$ is the product of the likelihood functions of each measurable in the oj segments. The denominator integrands apply to each segment. If the jth candidate consists of multiple segments then the X_j integral is replaced by the product of multiple integrals; one for each segment in the candidate set.

NUMERATOR — Λ_n

Consider first the numerator of (2.5). The state vector (X) consists of Geometric and Spectral (frequency) elements and hence will be expressed as $X = [X_G, X_S]$. The geometric state variables selected for long term (CZ-CZ) association are given by

$$X_G = [R_o\theta_o \; R_j\theta_j] \tag{2.6}$$

where,

$R_o\theta_o$ = Range and bearing at reference/current time (t_o)

$R_j\theta_j$ = Range and bearing that applies at the time of the most recent segment of the jth candidate set (t_j).

Using the form of Z given by (1.1) the numerator of the likelihood ratio as developed in A3 is given by

$$\boxed{\Lambda_n = \iint_{R_{oj}} P(X'_G)\, P(D_{oj}|X'_G)\, \underbrace{[\iint_{\theta_{oj}} P(Y_{oj}|D_{oj}\, X_G)\, d\theta_{oj}]}_{F_y}\, P(S_{oj}|D_{oj}\, X'_G)\, dR_{oj}} \tag{2.7}$$

where $R_{oj}\theta_{oj}$ are shorthand notation for R_oR_j and $\theta_o\theta_j$ respectively, primed state vector notation (X'_G) means that the state variables θ_o and θ_j have been replaced by the measured values ($\tilde{\theta}_o\tilde{\theta}_j$).

The terms of (2.7) are:

- $P(X'_G)$ — A priori pdf of target state (X'_G).
- $P(D_{oj}|X'_G)$ — Joint probability of detection of each segment conditioned on X'_G. This factors in the environment (propagation loss/convergence zones).
- F_y — The geometric measurement likelihood function given R_oR_j.
- $P(S_{oj}|D_{oj}\, X'_G)$ — The frequency spectrum likelihood function given X'_G.

Note that all of the non gaussian terms are outside the $\theta_o\theta_j$ integral thereby allowing an analytic evaluation of F_y. This is valid for the long term (CZ-CZ) association but not for short term association (e.g., within a single CZ).

For short term including mutual association, refer to A6.

DENOMINATOR — Λ_d

Now, consider the denominator of (2.5) designated by Λ_d. It is the product of terms Λ_{d_i} where the form of each term is identical except for the subscript identifying the segment (i=o,j).

Consider a typical oth segment having measurables $\theta \dot{\theta} \dot{f}$. As developed in A4,

$$\Lambda_{d_o} = P(\tilde{\theta}_o) \left[\int_{R_o} P(R_o)\, P(D_o|R_o)\, P(\dot{\theta}_o \dot{f}_o|D_o R_o)\, dR_o \right] P(S_o|D_o) \tag{2.8}$$

where $P(\tilde{\theta}_o)$ = A priori bearing state pdf evaluated at the measured bearing.

$P(\dot{\theta}_o \dot{f}_o|D_o R_o)$ = pdf of bearing rate and frequency rate conditional on detection and range. Since $\dot{\theta} \dot{f}$ (predicted) are a function of cross range rate ($\dot{X}$), in addition to R, it is computed by (o subscript is implied)

$$P(\dot{\theta} \dot{f}|DR) = \int_{\dot{X}} P(\dot{\theta} \dot{f}|DR\dot{X})\, P(\dot{X})\, d\dot{X} \tag{2.9}$$

COMPLETE Λ

Thus, Λ_n is given by (2.7) and Λ_d obtained as the product of terms like (2.8). Substitution into (2.5) yields the complete expected likelihood ratio given by (Π is product notation)

$$\boxed{\Lambda = \frac{\int_{R_o} \int_{R_j} P(X'_G)\, P(D_o D_j|X'_G)\, F_y\, \Lambda_S(X'_G)\, dR_o\, dR_j}{\prod_i \int_{R_i} P(R_i \theta_i)\, P(D_i|R_i)\, P(\dot{\theta}_i \dot{f}_i|D_i R_i)\, dR_i}} \tag{2.10}$$

where the expected likelihood ratio of the frequency spectrum data is given by

$$\Lambda_s(X'_G) = \frac{P(S_{oj}|D_{oj}\, A\, X'_G)}{P(S_{oj}|D_{oj}\, \overline{A}\, X'_G)} \tag{2.11}$$

Note that the denominator (normalizer) terms of (2.10) are relatively simple and are computed only once for each segment.

SECTION 3

TERMS OF EXPECTED LIKELIHOOD RATIO

State Prior

$P(X'_G)$ is the prior pdf of target state (geometric) with the $\tilde{\theta}_{0j}$ replacements. As developed in A5,

$$P(X'_G) = P(R_o)\, P(\tilde{\theta}_o)\, P(V\gamma)\cdot J \tag{3.1}$$

$P(R_o)$ and $P(\theta_o)$ are prior distributions of range and bearing. Typically these are assumed to be uniform distributions.

$P(V\gamma)$ is the prior pdf of target speed and heading where the arguments (V and γ) are computed from the state variables $(R_o\tilde{\theta}_o R_j\tilde{\theta}_j)$ and the inferred times $(t_o t_j)$. Assuming a uniform heading distribution ($P(\gamma) = \frac{1}{2\pi}$) and target speed normally distributed with mean $\tilde{V}$ and variance σ_V then,

$$P(V\gamma) = \frac{1}{2\pi} \cdot N\left(\frac{\tilde{V}-V}{\sigma_V}\right)^{(1)} \tag{3.2}$$

where V is computed from X'_G and the time difference (Δt_{oj}) as given in general by the vector equation ($\overline{D}_o$ is the displacement vector of sensor location at t_o relative to sensor location at t_j)

$$V = |\, \overline{D}_o + \overline{R}_o - \overline{R}_j \,| \,/\Delta t_{oj} \tag{3.3}$$

The Jacobian (J) transforms the $R_j\theta_j$ variables into $V\gamma$ variables as given by

$$J = \frac{R_j}{V\,\Delta t_{oj}^2} \tag{3.4}$$

Thus, $P(X'_G)$ (3.1) is easily computed for the measured values $(\tilde{\theta}_o\tilde{\theta}_j)$ and the assumed ranges $(R_o R_j)$.

(1) N(.) is notation for a normal distribution as given by

$$N\left(\frac{x}{\sigma_x}\right) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-1/2\left(\frac{x}{\sigma_x}\right)^2}$$

To visualize the effect of this prior state density term ($P(X'_G)$) within the R_oR_j integral of (2.10), consider the following as developed in B2. For every $V\gamma$ value (given $\tilde{\theta}_o\tilde{\theta}_j$) corresponding R_oR_j values can be computed, hence points in $V\gamma$ space can be mapped into R_oR_j space as illustrated in Figure 3-1.

Assume first that V is set at its mean value ($\tilde{V}$) and γ allowed to take on all values (0 to 2π). The resulting contour in R_oR_j space is an offset ellipse tilted at 45° as illustrated in Figure 3.1 (dashed ellipse). The contours for $\pm\sigma_V$ are also shown.

For the given bearings, the ellipse offset is proportional to own ship speed (V_o)(1) and the semi major/minor axes are proportional to target speed (V). Thus the 1σ $V\gamma$ distribution (also shown in Figure 3-1) maps into R_oR_j space as illustrated. Within the 1σ region of R_oR_j space, $P(V\gamma)$ will be high, decreasing to negligible values elsewhere.

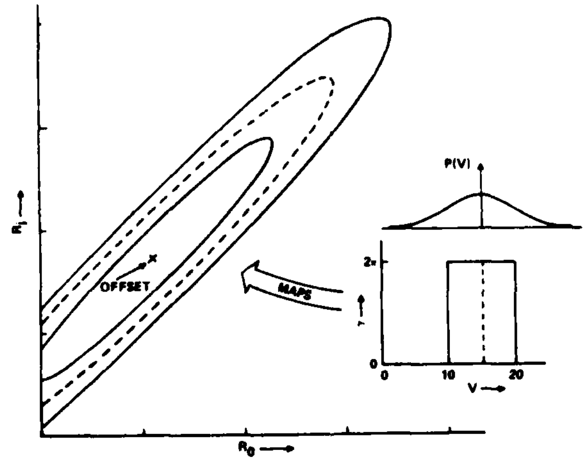

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Figure 3-1. Target State Prior in R_oR_j Space

Now consider the development of the second term in (2.10); the detection term.

DETECTION

$P(D_{oj} \mid X'_G)$ is the probability of detection occurring at times t_ot_j. Neglecting array gain variation with bearing, then probability of detection is a function of range only, so that

$$P(D_{oj} \mid X'_G) = P(D_o \mid R_o)\,P(D_j \mid R_oR_j) \qquad (3.5a)$$

(1) If the (o,j) measurements are made from different platforms (e.g., two ships or offboard arrays) then offset is proportional to platform separation

If more than a single candidate is included in the jth segment set, the additional segments (ℓ = 1,2,...) result in an additional factor; the probability of detection for these segments so that

$$P(D_{oj} \mid X'_G) = P(D_o \mid R_o)\, P(D_j \mid R_o R_j) \prod_{\ell} P(D_{\ell} \mid R_{\ell}(X'_G) \tag{3.5b}$$

where range (R_{ℓ}) is predicted from X'_G at the applicable times (t_{ℓ}).

As previously discussed, a target is tracked through a convergence zone and smoothed to obtain an estimate of Y_i (e.g. bearing, bearing rate and frequency rate). The time where the estimate applies is typically set at the track midpoint or endpoint. If this is the only time information used (total track time information ignored) than the P(D|R) function would be derived from the propagation loss/signal excess/probability of detection curves.

MEASUREMENTS

F_y is the measurement term of (2.10) as given by

$$F_y = \iint_{\theta_o \theta_j} P(Y_{oj} \mid D_{oj} X_G)\, d\theta_o \theta_j \tag{3.6}$$

Assuming the measurement errors are normally distributed, then

$$F_y = \sqrt{\frac{|W_y|}{(2\pi)^{N_y}}} \iint_{\theta_o \theta_j} e^{-\frac{1}{2}\Delta Y^T W_y \Delta Y}\, d\theta_o \theta_j \tag{3.7}$$

ΔY is the difference between the measured and true values; ΔY^T is the transpose of ΔY. Due to the conditioning (X_G) in (3.6) the true values are predicted from X_G.

W_y is the inverse of the diagonal measurement covariance matrix (M_y). M_y has elements such as $\sigma_{\theta_o}^2$, $\sigma_{\dot{\theta}_o}^2$, etc. Note that θ_o and $\dot{\theta}_o$ are uncorrelated with midpoint smoothing.

$|W_y|$ is the determinant of W_y.

N_y is the number of measurements.

As shown in C1, the solution to (3.7) is

$$\boxed{F_y = \underbrace{\sqrt{\frac{|W|}{(2\pi)^{N_1}}}}_{K_{Y1}} \; \underbrace{e^{-\frac{1}{2}\Delta Y_1^T W \Delta Y_1}}_{Q_y}} \quad (3.8)$$

where

$$\boxed{W = W_1 - (W_1H_1)(W_0 + H_1^T W_1 H_1)^{-1}(W_1H_1)^T} \quad (3.9)$$

ΔY_1 is the difference between the measured values (excluding $\tilde{\theta}_o\tilde{\theta}_j$) and those predicted from X'_G. Recall $X'_G = [R_o R_j \tilde{\theta}_o \tilde{\theta}_j]$.

W_0 is a diagonal 2 x 2 inverse covariance matrix with elements $(1/\sigma_{\theta_o}^2, 1/\sigma_{\theta_j}^2)$.

W_1 is a diagonal N_1 x N_1 inverse covariance matrix with reciprocal variance elements such as $(1/\sigma_{\dot{\theta}_o}^2, 1/\sigma_{\dot{f}_o}^2, 1/\sigma_{\dot{\theta}_j}^2, 1/\sigma_{\dot{f}_j}^2)$. Note the variables included are all the measurables except $\theta_o\theta_j$.

H_1 is the partial derivitive matrix of the Y_1 measurables with respect to $\theta_o\theta_j$ $(\partial\dot{\theta}_o\dot{f}_o\dot{\theta}_j\dot{f}_j/\partial\theta_o\theta_j)$.

N_1 is the number of measurables exceeding the two bearing measurement, i.e., $N_1 = N_y$ (the total number) − 2.

Note that the calculation of W given by (3.9) is fairly simple since the inverse is only 2 x 2.

Refer to C2 for an approximation to (3.8, 3.9) and C3 for an example.

It should be noted that the results given by (3.8) are general and apply to the association of more than a single past segment.

To illustrate the effect of this F_y term within the R_oR_j integral of (2.10), consider the following: Assume that $\dot{\theta}_o$ and $\dot{f}_o$ are measured in addition to $\theta_o\theta_j$. Using an approach similar to that of Figure 3-1 the 1σ $\dot{\theta}_o$ $\dot{f}_o$ contour maps into R_oR_j space (refer to B3) as illustrated in Figure 3-2.

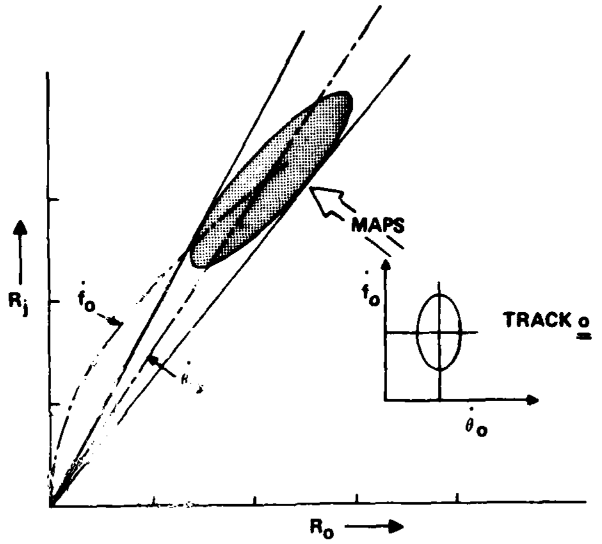


Figure 3-2. Rate Measurements in R_oR_j Space — Track o

Figure 3-3 shows the 1σ contour of $\dot{\theta}_o\dot{f}_o$ (from Figure 3-2) and also a corresponding contour for the jth segment.

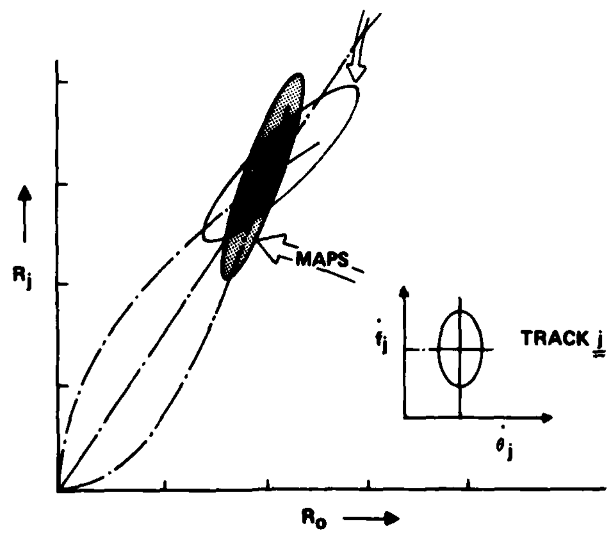


Figure 3-3. Rate Measurements in R_oR_j Space — Tracks o and j

The amount of overlap shown by the shaded regions is related to the magnitude of F_y, the conditional (R_oR_j) likelihood function of the geometric measurables.

FREQUENCY SPECTRUM LIKELIHOOD RATIO

The frequency spectrum likelihood ratio (Λ_S) is developed in Appendix D and E for two segments with spectra S_o,S_j. In the Λ formulation of (2.10) the spectrum likelihood ratio is conditioned on X'_G and hence Λ_S should be evaluated for all the R_oR_j combinations in the integral (summation) of (2.7). The formulation in Appendix D assumes that the spectra have been compensated for doppler shift (a function of range rate difference ($\Delta\dot{R}$)), where $\Delta\dot{R}$ is determined from X'_G. The formulation also assumes that the distribution of receiver amplitude for any given frequency cell is the same for each segment. This implies that the range dependence is ignored. A summary of the results of the frequency spectrum development is as follows.

The general result is given by

$$\Lambda_S = \left\{\underbrace{\prod^{n_2} [\Lambda_{S2}\lambda_{21} + \lambda_{22}]}_{\Lambda_{n2}}\right\} \left\{\underbrace{\prod^{n_1} [\Lambda_{S1}\lambda_{11} + \lambda_{12}]}_{\Lambda_{n1}}\right\} K_o \tag{3.10}$$

Π is product notation. n_2 is the number of frequency cells that match. A match is said to occur when a line in segment o and one in segment j appears at the same frequency (after doppler correction). n_1 is the number of cells where there is a mismatch (a line in only one segment). Λ_{S2} and Λ_{S1} are functions of the amplitude of the lines and the λ terms are constants derived from the a priori statistics.

Λ_{S2}, the "match" likelihood ratio, is given by

$$\Lambda_{S2} = \left(\frac{e^{.5P_F}}{2\sqrt{\pi}\,P_F}\right) e^{-1/2\left(\frac{\Delta S}{\sqrt{2}\sigma}\right)^2} e^{P_F\,\delta S/\sigma} \tag{3.11}$$

where

$\Delta S = (S_o - S_j)$ = the amplitude difference (db)

σ = amplitude fluctuation std deviation (dB)

δS = ave amplitude relative to threshold

P_F = probability of fade.

Λ_{S1}, the "mismatch" likelihood ratio, is given by

$$\Lambda_{S1} = \frac{1}{2 P_F} \; e^{-.8 \left(\frac{SE}{\sqrt{2}\,\sigma} \right)} \tag{3.12}$$

where

SE = amplitude relative to the threshold (dB).

These two Λ terms are illustrated in Figure 3-4.

The λ and K_o terms are a function of the a priori statistics of the target and extraneous lines as developed in Appendix D, E. Definitions of these statistics are

n_T = mean number of target lines.

n_E = mean number of extraneous lines

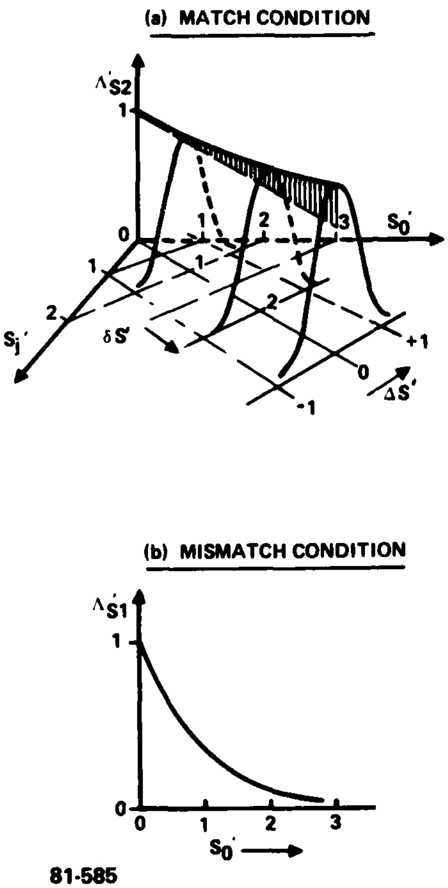
P_F = probability of fade

n_C = number of cells

and, $n_E' = n_E + (1\text{-}P_F/2)\, n_T$

Substituting the λ's from Table D-1 of D4 yields

$$\Lambda_S = \left[\prod^{n_2} \left[\Lambda_{S2} \left(\frac{(1\text{-}P_F)\, n_T\, n_C}{n_E'^{\,2}} \right) + 1 \right] \bullet \prod^{n_1} \left[\Lambda_{S1} \left(\frac{P_F\, n_T}{2 n_E'} \right) + \frac{n_E}{n_E'} \right] \right] e^{n_T (1\text{-}P_F)} \tag{3.13}$$



NORMALIZING CONSTANTS

$$\Lambda_{S2} = \Lambda_{S2}' \left[\frac{e^{.5 P_F}}{2\sqrt{\pi} P_F} \right]$$

$$\Lambda_{S1} = \Lambda_{S1}' \left[\frac{.5}{P_F} \right]$$

$$S_0' = \frac{S_0 \cdot T}{\sqrt{2}\,\sigma}$$

$$S_j' = \frac{S_j \cdot T}{\sqrt{2}\,\sigma}$$

$$\delta S' = \frac{S_{ave} \cdot T}{\sigma}$$

$$\Delta S' = \frac{S_0 - S_j}{\sqrt{2}\,\sigma}$$

Figure 3-4. Frequency Spectrum Likelihood Ratio Components

If amplitude data is not made available, the spectrum is referred to as "clipped", then only the frequency cells containing a detection are available. For this case, Λ_{S2} and Λ_{S1} are unity and hence the products (e.g., $\prod^{n_2}(\ )$) can be replaced by the exponent form (e.g., $(\ )^{n_2}$) so that

$$\Lambda_S = \left[\frac{(1 - P_F)\, n_T n_C}{n_E'^{\,2}} + 1\right]^{n_2} \left[\frac{P_F n_T + 2n_E}{2n_E'}\right]^{n_1} e^{n_T(1-P_F)} \tag{3.14}$$

Consider a typical case where $P_F = .2$, $n_T = 2$, $n_E = 2$ and $n_C = 400$ ($n_E' = 3.8$).

then,

$$\Lambda_S = (44 + 1)^{n_2} (0.05 + 0.53)^{n_1}\, 6$$

or (3.15)

$$\text{Log}\, \Lambda_S = 1.65\, n_2 - 0.24\, n_1 + 0.78$$

This function is plotted in Figure 3-5.

Another approach was initially used to develop the spectrum likelihood ratio (Appendix F). It was dropped in favor of the approach of Appendix D basically for two reasons; amplitude information is not utilized (clipping is assumed) and no extraneous lines are included.

DENOMINATOR TERMS (Reference (2.10))

- $P(R\theta)$ — State pdf R and θ are considered essentially independent so that

$$P(R_i\theta_i) = P(R_i)\, P(\theta_i) \tag{3.16}$$

- $P(D|R)$ — Same as numerator detection terms.

- $P(\dot{\theta}\dot{f}|DR)$ — As shown in A4, this pdf is obtained by including relative cross range rate ($\dot{X}$) and integrating it out as given by

$$P(\dot{\theta}\dot{f}|DR) = \int_{\dot{X}} P(\dot{\theta}\dot{f}|DR\dot{X})\, P(\dot{X})\, d\dot{X}$$

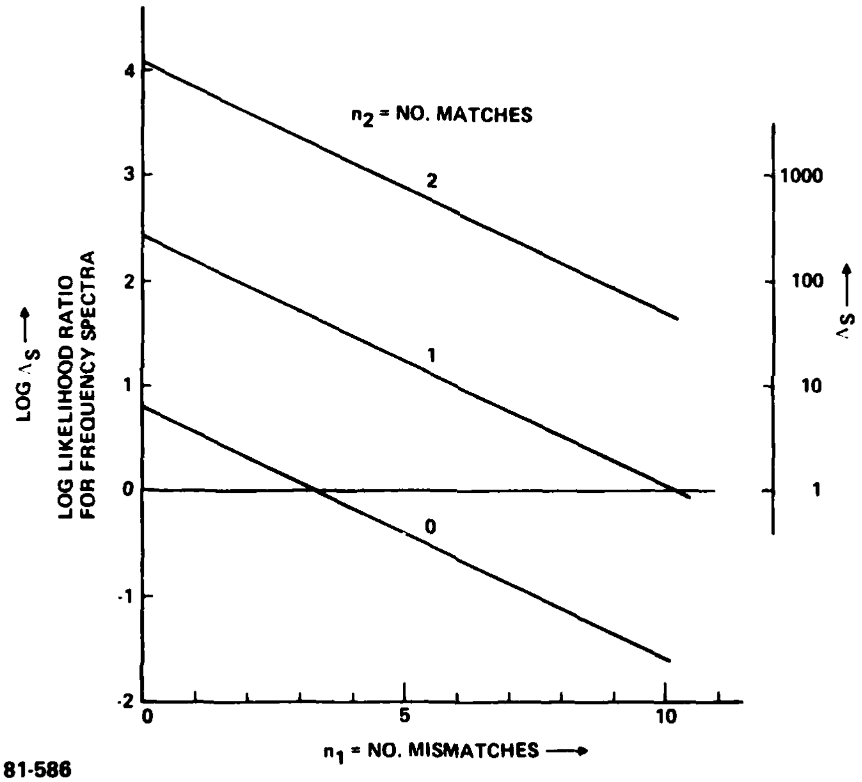


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Figure 3-5. Log Likelihood Ratio for Frequency Spectra

Assuming normally distributed $\dot{\theta}\dot{f}$ errors with variances $(\sigma_{\dot{\theta}}^2\ \sigma_{\dot{f}}^2)$ then,

$$P(\dot{\theta}\dot{f}|DR\dot{X}) = N\left(\frac{\dot{\theta} - \dot{X}/R}{\sigma_{\dot{\theta}}}\right)N\left(\frac{\lambda\dot{f} - \dot{X}^2/R}{\sigma_{\dot{f}}}\right) \tag{3.17}$$

Assuming normally distributed a priori target speed with mean $\tilde{V}$ and variance σ_V^2 and uniformly distributed target heading then,

$$P(\dot{X}) = \frac{1}{\pi}\int_{V=|\dot{X}_T|}\left[N\left(\frac{V-\tilde{V}}{\sigma_V}\right)\frac{1}{\sqrt{V^2 - \dot{X}_T^2}}\right] dV \tag{3.18}$$

where $\dot{X}_T$ is target cross range rate ($\dot{X}_T = \dot{X} + \dot{X}_o$), $\dot{X}_o$ is own-ship cross range rate.

This function is shown in Figure 3-6 for V = 15 kts for $\dot{X}_o = 0$. For finite $\dot{X}_o$, simply shift the curves left by an amount $\dot{X}_o$.

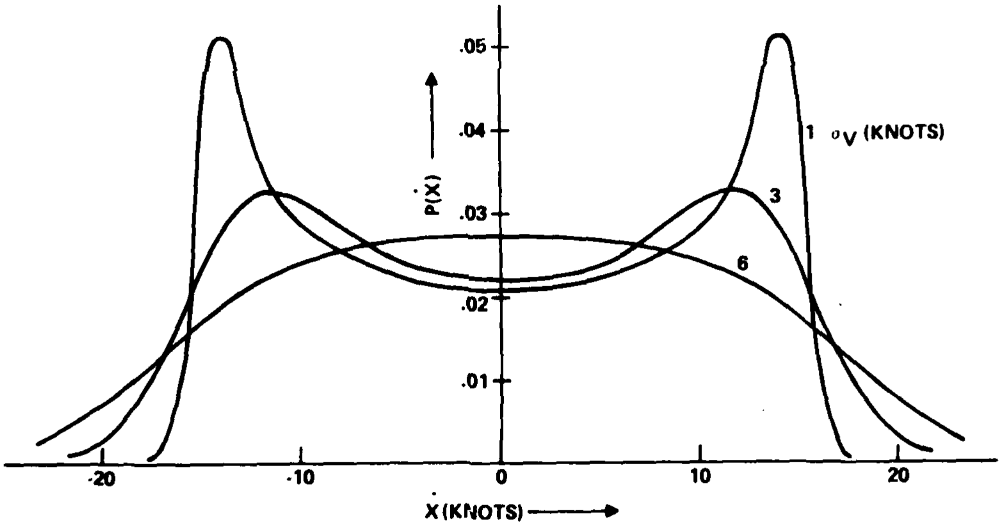

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Figure 3-6. Cross Range Rate Prior Distribution

SECTION 4

LIKELIHOOD SUMMARY

This basic Λ expression and the terms included are summarized below. (Refer to Appendix A.)

$$\Lambda = \frac{\iint_{R_o R_j} P(X'_G) \cdot P(D_{oj} \mid X'_G) \cdot F_y \cdot \Lambda_S(X'_G)\, d\, R_o R_j}{\prod_i \int_{R_i} P(R_i \tilde{\theta}_i) \cdot P(D_i \mid R_i) \cdot P(Y_{1_i} \mid D_i R_i)\, dR_i} \qquad (2.10)$$

Target State Prior

$$P(X'_G) = P(R_o) \cdot P(\tilde{\theta}_o) \cdot P_{V_\gamma} J \qquad (3.1)$$

Detection

$$P(D_{oj} \mid X'_G) = P(D_o D_j \mid R_o R_j) \cdot \prod P(D_{\cdot} \mid X'_G) \qquad (3.5b)$$

Measurements (Geometry)

$$F_y = K_{Y1}\, e^{-\Delta Y_1^T W \Delta Y_1} \qquad (3.8)$$

Measurement (Spectra)

$$\Lambda_S(X'_G) = \prod^{n_2} [\Lambda_{S2}\, \lambda_{21} + 1] \prod^{n_1} [\Lambda_{S1}\, \lambda_{11} + \lambda_{12}] \cdot e^{n_T (1-P_F)} \qquad (3.10)$$

Mapping of the a priori speed/heading, detection and measurement densities into range ($R_o R_j$) space as previously discussed is shown as a composite in Figure 4-1. Note that bearing rate and frequency rate are the only measurements included here. Normalization (Λ_d), the R_j factor of (3.4) and the frequency spectrum term are not included in this illustration.

The association measure Λ is proportional to the summation, over the $R_o R_j$ space, of the product of terms in (2.10). Thus only the regions of overlap of all three terms of Figure 4-1 will contribute to the sum. Note that the primary contribution is from the zones at $R_o R_j$ = 60,90; the true range pair.

A block diagram showing the process used to compute Λ_{oj} and F_{oj} is shown in Figure 4-2.

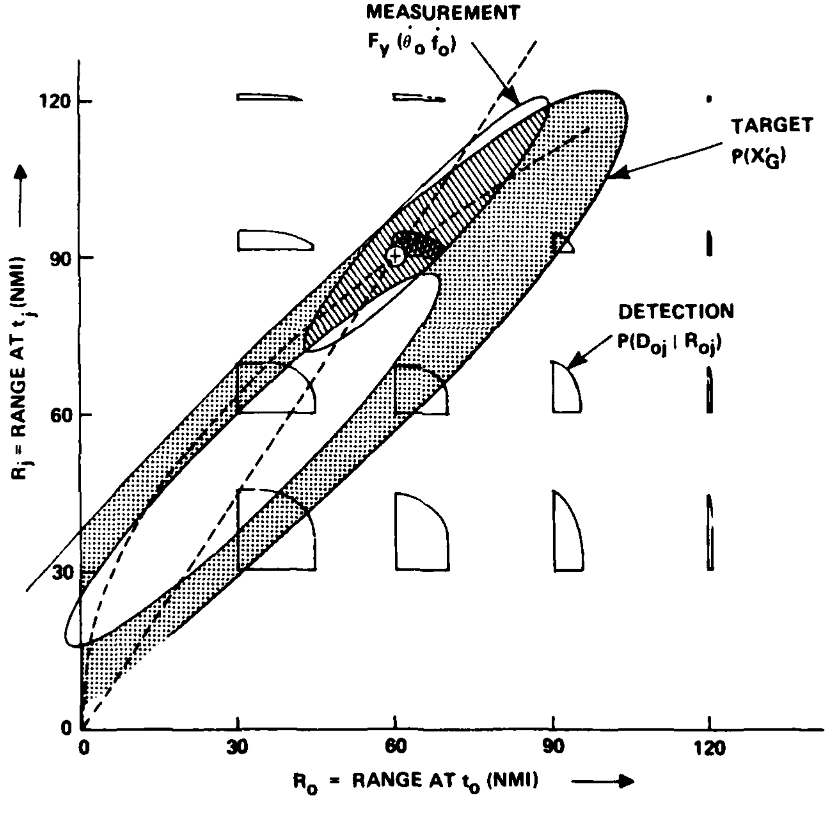

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Figure 4-1. Composite Mapping into R_oR_j Space

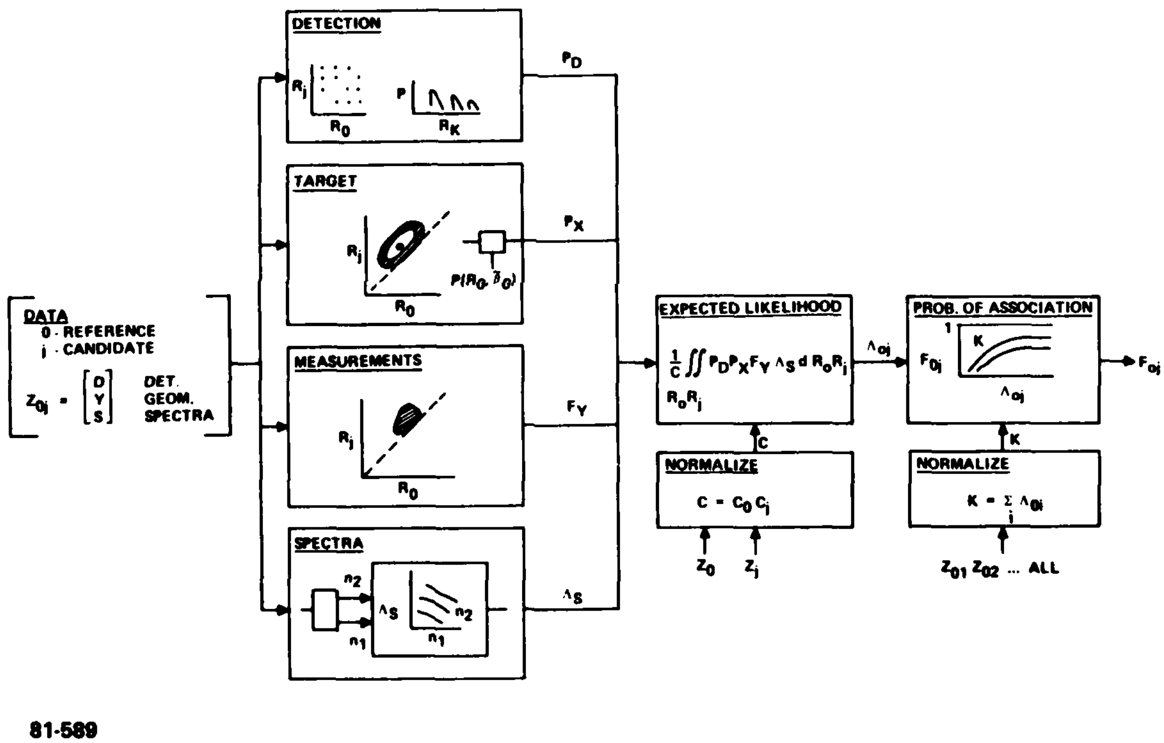


Figure 4-2. Data Association Technique

APPENDIX A — ASSOCIATION MEASURE AND LIKELIHOOD RATIO DEVELOPMENT

A1 - DERIVATION OF ASSOCIATION MEASURE $P(A_{oj} \mid Z_{ojk})$

The general problem is to determine how well a current/reference segment (subscript o) associates with one or more candidate segments (subscript j). The remaining segments are identified with subscript k so that the total data set is defined by Z_{ojk}. Thus the probability of association of the current (o) and candidates (j) given all the data is expressed by

$$F_{oj} = P(A_{oj} \mid Z_{ojk}) \tag{1}$$

Note that the j and k segment sets consist of one or more segments.

Applying Bayes' rule yields

$$P(A_{oj} \mid Z_{ojk}) = \frac{P(Z_{ojk} \mid A_{oj})\, P(A_{oj})}{P(Z_{ojk})} \tag{3}$$

Consider the first numerator term. The k subscripted segments are not conditioned by A_{oj} thus

$$P(Z_{ojk} \mid A_{oj}) = P(Z_{oj} \mid A_{oj}) \cdot P(Z_k) \tag{4}$$

Consider the following simple equality

$$P(Z_k) = \frac{P(Z_o) \cdot [P(Z_{j1}) \cdot P(Z_{j2}) \ldots]}{P(Z_o) \cdot [P(Z_{j1}) \cdot P(Z_{j2}) \ldots]} \cdot P(Z_k) \tag{5}$$

Note that the numerator is

$$P(Z_o, Z_j, Z_k \mid \overline{A}_{ojk}) \triangleq P(Z_{ojk} \mid \overline{A}_{ojk}) \tag{6}$$

Where $\overline{A}_{ojk}$ is the condition that the ojk (i.e., all) segments are not associated.

Also note that the denominator is

$$P(Z_o Z_j \mid \overline{A}_{oj}) \triangleq P(Z_{oj} \mid \overline{A}_{oj}) \tag{7}$$

Thus (5) can be written as

$$P(Z_k) = \frac{P(Z_{ojk} \mid \overline{A}_{ojk})}{P(Z_{oj} \mid \overline{A}_{oj})} \tag{8}$$

Substituting into (4) yields

$$P(Z_{ojk} \mid A_{oj}) = \frac{P(Z_{oj} \mid A_{oj})\, P(Z_{ojk} \mid \overline{A}_{ojk})}{P(Z_{oj} \mid \overline{A}_{oj})} \tag{9}$$

Then substitution into (3) yields

$$P(A_{oj} \mid Z_{ojk}) = \underbrace{\left[\frac{P(Z_{oj} \mid A_{oj})}{P(Z_{oj} \mid \overline{A}_{oj})}\right]}_{\Lambda_{oj}} \frac{P(Z_{ojk} \mid \overline{A}_{ojk})\, P(A_{oj})}{P(Z_{ojk})} \tag{10}$$

Note that the first ratio is the expected likelihood ratio (Λ_{oj}) and the last term the a priori probability of association of the oj segments. Since subscripts ojk indicate all the segments, it may be clearer to drop them so that

$$P(A_{oj} \mid Z) = \frac{\Lambda_{oj}\, P(A_{oj})}{P(Z) \,/\, P(Z \mid \overline{A})} \tag{11}$$

where

$$\Lambda_{oj} = \frac{P(Z_{oj} \mid A_{oj})}{P(Z_{oj} \mid \overline{A}_{oj})} \tag{12}$$

If the past candidate (subscript j) consists of only a single segment then the denominator of (11) can be formulated as follows.

For m past contacts of which none or one are associated with the current (o) segment then

$$P(Z) = P(Z \mid \overline{A})\, P(\overline{A}) + \sum_{i}^{m} P(Z \mid A_{oi})\, P(A_{oi}) \tag{14}$$

Substitution into (11) yields

$$P(A_{oj} \mid Z) = \frac{\Lambda_{oj}}{\dfrac{P(\overline{A})}{P(A_{oj})} + \sum\limits_{i}^{m} \dfrac{P(Z \mid A_{oi})\, P(A_{oi})}{P(Z \mid \overline{A})\, P(A_{oj})}} \tag{15}$$

Note that

$$P(Z \mid A_{oi}) = P(Z_{oi} \mid A_{oi}) \frac{P(Z \mid \overline{A})}{P(Z_{oi} \mid \overline{A}_{oi})} \tag{16}$$

$$= \Lambda_{oi}\, P(Z \mid \overline{A})$$

Assume that all $P(A_{oi})$ are equal and substitute (16) into (15) to obtain

$$P(A_{oj} \mid Z) = \frac{\Lambda_{oj}}{\dfrac{P(\overline{A})}{P(A_{oj})} + \sum\limits_{i}^{m} \Lambda_{oi}} \tag{17}$$

Assuming a diffuse prior where P_g is the probability that the correctly associated past segment exists in the set of m segments. Then $P(\overline{A})$ is the probability that it does not exist as given by $(1\text{-}P_g)$. $P(A_{oj})$ is the probability that the j th segment is associated with the o segment as given by $P_g \cdot 1/m$ so that the first denominator term is $((1\text{-}P_g) / P_g)$ m. Thus,

$$P(A_{oj} \mid Z) = \frac{\Lambda_{oj}}{\left(\dfrac{1-P_g}{P_g}\right) m + \sum\limits_{i}^{m} \Lambda_{oi}} \tag{18}$$

<u>A2 – Λ FORMULATION</u>

The oj subscript on Λ will be dropped for simplicity, from A1(12)

$$\Lambda = \frac{P(Z_{oj} \mid A_{oj})}{P(Z_{oj} \mid \overline{A}_{oj})} \tag{1}$$

Since segments o and j are unassociated (denominator) they are independent so that

$$P(Z_{oj} \mid \overline{A}_{oj}) = P(Z_o)\, P(Z_j) \tag{2}$$

These density functions can be evaluated by forming the joint measurement/state density function and then integrating out the state variable so that (association condition implied in each integrand)

$$\Lambda = \frac{\int_X P(Z_{oj}X)\, dX}{\int_{X_o} P(Z_oX_o)\, dX_o \int_{X_j} P(Z_jX_j)\, dX_j} \tag{3}$$

Since $P(Z_{oj}\, X) = P(Z_{oj} \mid X)\, P(X)$ and similarly for $P(Z_oX)$ and $P(Z_jX)$ then,

$$\boxed{\Lambda = \frac{\int_X P(Z_{oj} \mid X)\, P(X)\, dX}{\int_{X_o} P(Z_o \mid X_o)\, P(X_o)\, dX_o \int_{X_j} P(Z_j \mid X_j)\, P(X_j)\, dX_j}} \tag{4}$$

A3- Λ_n FORMULATION

Λ_n is defined as the numerator of Λ (refer to A2 (4)). Thus,

$$\Lambda_n = \int_X P(Z_{oj} \mid X)\, P(X)\, dX \tag{1}$$

where X consists of geometric (X_G) and spectral (X_S) state vector components ($X = [X_G\, X_S]$).

The measurement vector consists of detection (D), geometric (Y) and spectrum (S) elements ($Z_{oj} = [DYS]_{oj}$) (refer to Section 2 (1.1)). Expanding the first intergrand term yields

$$P(Z_{oj} \mid X) = P(DYS \mid X) \tag{2}$$

$$= P(D \mid X_GX_S)\, P(Y \mid DX_GX_S)\, P(S \mid DX_GX_S)$$

where subscripts (oj) apply for DY and S. The detection event is conditioned on source spectra (X_S), a fairly complex relationship hence the X_S conditioning will be removed from the first term. The X_S condition is irrelevent in the measurement (2nd) term and hence will be dropped.

Then, substitution into (1) yields

$$\Lambda_n = \int_{X_G}\int_{X_S} P(D|X_G)\, P(Y|DX_G)\, P(S|DX_GX_S)\, P(X_G)\, P(X_S)\, dX_SX_G$$

$$= \int_{X_G} P(X_G)\, P(D|X_G)\, P(Y|DX_G)\, \underbrace{\left[\int_{X_S} P(X_S)\, P(S|DX_GX_S)\, dX_S\right]}_{P(S|DX_G)} dX_G \qquad (3)$$

The geometric state variables selected are range and bearing for both the current (o) and the most recent past segment (j) so that

$$X_G = [R_o\theta_oR_j\theta_j] \qquad (4)$$

where the corresponding times (t_ot_j) are implied.

Bearings $(\theta_o\theta_j)$ are always measured. (~) will be used to identify measured values.

Thus, from (3)

$$\Lambda_n = \int_{X_G} P(X_G)\, P(D|X_G)\, P(Y|DX_G)\, P(S|DX_G)\, dX_G \qquad (5)$$

Since $P(X_G)$ is a weak function of $\theta_o\theta_j$, a good approximation[1] is to assume that the state variables $\theta_o\theta_j$ equal the measured values $(\tilde{\theta}_o\tilde{\theta}_j)$. Prime (′) will be used to indicate this substitution, i.e., $P(X_G) \rightarrow P(X'_G)$. The same applies to the detection and spectrum density functions. Thus all except the measurement density $P(Y|DX_G)$ can be moved outside the $\theta_o\theta_j$ integral so that

$$\boxed{\Lambda_n = \int\int_{R_oR_j} P(X'_G)\, P(D|X'_G)\, \underbrace{\left[\int\int_{\theta_o\theta_j} P(Y|DX_G)\, d\theta_o\theta_j\right]}_{F_y} P(S|DX'_G)\, dR_oR_j} \qquad (6)$$

(1) This is a good approximation for temporal association where time between segments is large (> 30 minutes) as it is for segments in different convergence zones. For short times another formulation applies (Refer to A6).

A4 – Λ_d DEVELOPMENT

Λ_d is defined as the denominator of Λ (Refer to (2.4a). The state variables selected are range, bearing, range rate and cross range rate. Thus for a typical o th segment

$$X_G = R_o \theta_o \dot{R}_o \dot{X}_o \tag{1}$$

The spectrum component of the state vector is X_S, thus for a typical segment (similar to A3(3))

$$\Lambda_{d_o} = \int\int_{X_G X_S} P(D_o|X_G)\, P(Y_o|D_o X_G)\, P(S_o|D_o X_G X_S)\, P(X_G)\, P(X_S)\, dX_G X_S \tag{2}$$

Assuming detection is a weak function of bearing (omni array gain) then

$$P(D_o|X_G) = P(D_o|R) \tag{3}$$

Assuming that the measurables are bearing, bearing rate and frequency rate $(\tilde{\theta}_o \tilde{\dot{\theta}}_o \tilde{\dot{f}}_o)$ then, with o subscripts implied,

$$P(Y \mid D\, X_G) = P(\tilde{\theta} \mid D\, \theta)\, P(\dot{\theta}\ \dot{f} \mid D\, R\dot{X}) \tag{4}$$

Note that $\dot{R}$ is not included since $\dot{\theta}$ $\dot{f}$ are a function of R and $\dot{X}$ only; not $\dot{R}$.

The spectrum is only a function of range; doppler shift for a single segment is irrelevent so that

$$P(S \mid D\, X_G) = P(S \mid D\, R) \tag{5}$$

The state variables are essentially independent so that

$$P(X_G) = P(R)\, P(\theta)\, P(\dot{R})\, P(\dot{X}) \tag{6}$$

Substitution for these four functions into (2) and separation of integrals yields

$$\Lambda_{d_o} = \left[\int_\theta P(\tilde{\theta}\,|\,\theta)\,P(\theta)\,d\theta\right]$$

$$\cdot \left\{ \int_R P(D\,|\,R)\left[\int_{X_S} P(S\,|\,D\,RX_S)\,P(X_S)\,dX_S\right] \right. \tag{7}$$

$$\left. \cdot \underbrace{\left[\int_X P(\dot{\theta}\,\dot{f}\,|\,DR\dot{X})\,P(\dot{X})\,d\dot{X}\right]}_{P(\dot{\theta}\,\dot{f}\,|\,DR)} P(R)\,dR \right\} \cdot \underbrace{\int_{\dot{R}} P(\dot{R})\,d\dot{R}}_{\text{unity}}$$

The first integral reduces to the value of $P(\theta_o)$ (bearing prior) evaluated at the measured value ($\theta = \tilde{\theta}$) and the last is unity. Thus with the indicated definitions and ignoring the range dependence (condition) of the spectrum density

$$\boxed{\Lambda_{d_o} = P(\tilde{\theta}_o)\,P(S_o|D_o)\int_{R_o} P(D_o|R_o)\,P(\dot{\theta}_o\dot{f}_o\,|\,D_oR_o)\,P(R_o)\,dR_o} \tag{8}$$

The expression for other segments has the same form, simply replace subscript o with the appropriate segment subscript.

A5 - PRIOR STATE — $P(X'_G)$

The prior pdf of the geometric range, bearing state vector at the time of segments o and j is given by

$$P(X'_G) = P(R_o\tilde{\theta}_oR_j\tilde{\theta}_j) \tag{1}$$

Range and bearing densities are essentially independent so that

$$P(X'_G) = P(R_o)\,P(\tilde{\theta}_o)\,P(R_j\tilde{\theta}_j\,|\,R_o\tilde{\theta}_o) \tag{2}$$

The third term can be evaluated from the known target speed and heading density $P(V\ \gamma)$ using the Jacobian (J) transformation as given by

$$P(R_j \tilde{\theta}_j \mid R_o \tilde{\theta}_o) = P(V\ \gamma) \cdot J \tag{3}$$

where

$$J = \left| \frac{\partial V \gamma}{\partial R_j \theta_j} \right| = \begin{vmatrix} \frac{\partial V}{\partial R_j} & \frac{\partial V}{\partial \theta_j} \\ \frac{\partial \gamma}{\partial R_j} & \frac{\partial \gamma}{\partial \theta_j} \end{vmatrix} = \begin{vmatrix} \frac{\cos \phi}{\Delta t} & \frac{R_j}{\Delta t} \sin \phi \\ -\frac{\sin \phi}{V\ \Delta t} & \frac{R_j}{V\ \Delta t} \cos \phi \end{vmatrix} \tag{4}$$

where

$|\ |$ is determinant notation.

ϕ is target heading relative to the line of bearing (θ_j).

thus,

$$J = \frac{R_j}{V\ \Delta t^2} \tag{5}$$

Note that target speed (V) is a function of $(R_o R_j \tilde{\theta}_o \tilde{\theta}_j)$ as given by

$$V = |\bar{D}_o + \bar{R}_o - \bar{R}_j| \,/\, \Delta t \tag{6}$$

where

$||$ is magnitude notation and $\bar{R}$ is vector notation. $\bar{D}_o$ is the displacement vector of sensor location at t_o relative to sensor location at t_j. $\Delta t = t_o - t_j$.

A6 – SHORT TERM ASSOCIATION

In the previous analysis the state prior could be removed from inside the $\theta_o \theta_j$ integral since the time difference between segments was assumed to be large. This was an important simplification since the $\theta_o \theta_j$ integrand was then gaussian and hence readily integrable. This assumption is valid for CZ to CZ temporal association however it breaks down for short term or for mutual (simultaneous) association.

This analysis develops the short term likelihood expressions using a different state vector.

In the long term temporal association formulation X_G was defined as the range/bearing at the times of the two segments as given by A3 (4).

For short term association a more natural state vector is

$$X_G = [R_o \theta_o V \gamma]$$

where $V\gamma$ are target speed and heading; V is assumed to be gaussian with mean $\tilde{V}$ and variance σ_V^2. Then A3(5) can be written as

$$\Lambda_n = \iint_{R_c \gamma} \underbrace{P(D|R_o \tilde{\theta}_o \tilde{V} \gamma)}_{X''_G} P(S|DX''_G)\, P(R_o \tilde{\theta}_o \gamma) \underbrace{\left[\iint_{\theta_o V} P(V)\, P(Y|X_G)\, d\theta_o V\right]}_{F'_y} dR_o \gamma \qquad (1)$$

Note that the detection and spectrum densities have been removed from the $\theta_o V$ integral since they are weak functions of θ_o and V and hence the measured value of $\theta_o (\tilde{\theta}_o)$ and the a priori (mean) value of target velocity ($\tilde{V}$) are used.

For comparison the previous $\theta_o\theta_j$ integral and the $\theta_o V$ integral are rewritten below

- Long term (refer to C1(1))

$$F_y = \iint_{\theta_o\theta_j} P(Y|X_G)\, d\theta_{o}\theta_j = \iint_{\theta_o\theta_j} P(\tilde{\theta}_o|\theta_o) \cdot P(\tilde{\theta}_j|\theta_j)\, P(Y_1|X_G)\, d\theta_o\theta_j \tag{2}$$

- Short term

$$F_y' = \iint_{\theta_o V} P(\tilde{V}|V)\, P(Y|X_G)\, d\theta_o V = \iint_{\theta_o V} P(\tilde{\theta}_o|\theta_o)\, P(\tilde{V}|V)\, P(Y_1'|X_G)\, d\theta_o V \tag{3}$$

Note that the forms are the same except that θ_j is replaced by V and that Y_1' includes all measurables except θ_o (recall Y_1 excluded both θ_o and θ_j).

Referring to (3.6) and its solution (3.8) the solution to the $\theta_o V$ integral is given by

$$F_y' = \sqrt{\frac{|W'|}{(2\pi)^{N_1'}}}\; e^{-\frac{1}{2}\Delta Y_1' W' \Delta Y_1'} \tag{4}$$

where

$$W' = W_1' - (W_1' H_1')(W_0' + H_1'^T W_1' H')^{-1} (W_1' H_1')^T \tag{5}$$

$\Delta Y_1'$ is the difference between the measured values (excluding θ_o) and those predicted from X_G''.

where

$$X_G'' = [R_o \tilde{\theta}_o \tilde{V} \gamma]$$

W_0' is a diagonal 2 x 2 inverse covariance matrix with elements $(1/\sigma_{\theta_o}^2, 1/\sigma_V^2)$.

W_1' is a diagonal N_1' x N_1' inverse covariance matrix with elements such as $(1/\sigma_{\theta_j}^2\ 1/\sigma_{\dot{\theta}_o}^2, \ldots)$.

Note the variables included are all the measurables except θ_o.

H_1' is the partial derivitive matrix of the Y_1' measurables with respect to θ_o and V.

N_1' is the number of measurables excluding θ_o. Thus if N_y is the total number $N_1' = N_y - 1$.

For comparison, the previous (long term) numerator likelihood ratios and the short term expression (1) are rewritten below

- Long Term

$$\Lambda_n = \int_{R_o}\int_{R_j} \underbrace{P(R_o\tilde{\theta}_o R_j\tilde{\theta}_j)}_{X_G'} \cdot P(D|X_G')\, P(S|DX_G') \cdot F_y\, dR_o R_j \tag{6}$$

- Short Term

$$\Lambda_n = \int_{R_o}\int_{\gamma} P(R_o\tilde{\theta}_o\gamma) \cdot P(D|X_G'') \cdot P(S|DX_G'') \cdot F_y'\, dR_o\gamma \tag{7}$$

The variables of the prior term are essentially independent so that

$$P(R_o\tilde{\theta}_o\gamma) = P(R_o)\, P(\tilde{\theta}_o)\, P(\gamma) \tag{8}$$

Again assuming probability of detection to be a soft function of azimuth then

$$P(D|X_G'') = P(D_o|R_o)\, P(D_j|R_j(X_G'')) \tag{9}$$

The spectrum term is a function of the change in range rate between the two segments so that

$$P(S|DX_G'') = P(S|D\Delta\dot{R}(X_G'')) \tag{10}$$

Thus the short term expression is

$$\boxed{\Lambda_n = P(\tilde{\theta}_o) \int_{R_o} P(D_o|R_o)\, P(R_o) \int_{\gamma} P(D_j|X_G'')\, P(\gamma)\, P(S|DX_G'')\, [F_y']\, dR_o\gamma} \tag{11}$$

The denominator (Λ_d) is the same fo. the short term and long term formulations.

APPENDIX B — GEOMETRY AND R_oR_j CONTOURS

B1 — $\dot{\theta}\dot{t}$ MODEL

Nonlinear Own Ship Motion/Two Platforms

Consider the general case where the o and j th measurements are made from different platforms (2 ships) or from the same ship with displacement caused by own ship motion. In either case D_o is the displacement between the position of the two platforms (ships) for the time interval $\Delta t = (t_o - t_j)$ and ϕ is the "heading" of the displacement line (vector).

The geometry is shown in Figure B1.

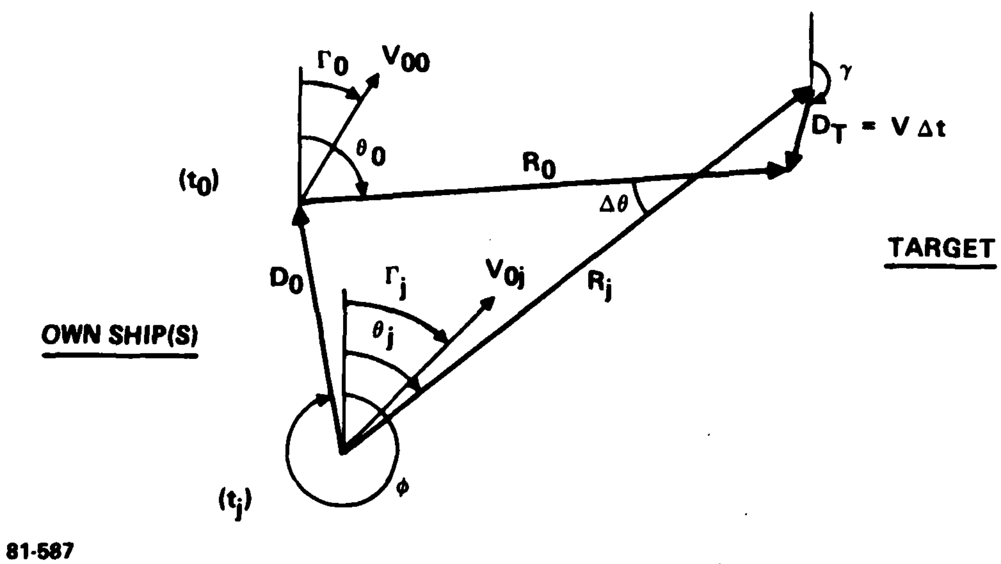


Figure B1. Geometry

Definition of terms:

Δt = $t_o - t_j$ = time interval between measurements

D_o, D_T = own ship and target displacements during Δt

ϕ = "heading" of own ship displacement line/vector

γ, V = target course (heading) and speed

V_{oo}, Γ_o = own ship speed and heading at time t_o

V_{oj}, Γ_j = own ship speed and heading at time t_j

θ_o, θ_j = true bearing at times t_o and t_j

$\Delta\theta$ = $\theta_o - \theta_j$ = bearing difference

R_o, R_j = Range at t_o, t_j

Solving for the relative bearing rates at t_o and t_j yields

$$\dot{\theta}_o = \frac{1}{R_o}\left\{\frac{R_j \sin \Delta\theta - D_o \sin(\theta_o - \phi)}{\Delta t} + V_{oo} \sin(\theta_o - \Gamma_o)\right\} \tag{1}$$

$$\dot{\theta}_j = \frac{1}{R_j}\left\{\frac{R_o \sin \Delta\theta - D_o \sin(\theta_j - \phi)}{\Delta t} + V_{oj} \sin(\theta_j - \Gamma_j)\right\}$$

Range acceleration ($\ddot{R}$) equals ($-\lambda\dot{f}$) where λ is wavelength. $\ddot{R}$ also equals $\dot{\theta}^2 R$ so that frequency rate is given by

$$\lambda\dot{f}_o = -\dot{\theta}_o^2 R_o$$

and

$$\lambda\dot{f}_j = -\dot{\theta}_j^2 R_j \tag{2}$$

Linear Own Ship Motion

Assuming own ship is on a constant course and speed between points o and j, then $\phi = \Gamma_o = \Gamma_j$ and $D_o/\Delta t = V_{oo} = V_{oj}$. Then (1) and (2) reduce to

$$\begin{aligned} \dot{\theta}_o &= \frac{R_j}{R_o}\,\frac{\sin\Delta\theta}{\Delta t}, \quad \dot{\theta}_j = \frac{R_o}{R_j}\,\frac{\sin\Delta\theta}{\Delta t} \\ \lambda\dot{t}_o &= -\frac{R_j^2}{R_o}\left(\frac{\sin\Delta\theta}{\Delta t}\right)^2, \quad \lambda\dot{t}_j = -\frac{R_o^2}{R_j}\left(\frac{\sin\Delta\theta}{\Delta t}\right)^2 \end{aligned} \tag{3}$$

B2 — TARGET SPEED CONTOURS ON A R_oR_j PLOT

The problem here is to determine how the prior target speed/heading uncertainty area maps into R_oR_j space. The inputs/givens are own ship displacement ($D_o\ \phi$), the two bearings ($\theta_o\theta_j$) and Δt. Thus all lines in Figure B1 are known/fixed except the target displacement vector ($D_T\gamma$) or ($V\Delta t, \gamma$). For a fixed/assumed value of target speed ($D_T/\Delta t$) variations in target heading will generate the desired range pairs (R_oR_j).

The approach is to solve the two triangles of Figure B2 (also refer to Figure B1).

The range pair at the bearing crossover point 1 is obtained using the law of sines so that

$$\begin{aligned} R_{so} &= D_o \sin(\theta_j - \phi)/\sin\Delta\theta \\ R_{sj} &= D_o \sin(\theta_o - \phi)/\sin\Delta\theta \end{aligned} \tag{1}$$

Note in passing that this would be the crossfix range solution(s) assuming zero target speed.

The true range(s) (R_oR_j) differ from this crossover range ($R_{so}R_{sj}$) by the increment $\Delta R_o\ \Delta R_j$ due to target motion as shown. Using the law of cosines yields the relationship between ΔR_o and ΔR_j given the $\Delta\theta$ measurement and the assumed target speed/displacement (D_T) so that

$$D_T^2 = \Delta R_o^2 + \Delta R_j^2 - 2\,\Delta R_o\,\Delta R_j \cos\Delta\theta \tag{2}$$

Thus given an assumed target speed and Δt then D_T is known and ΔR_j can be solved as a function of ΔR_o. The result is a contour in $R_o R_j$ space as illustrated in Figure B3. Rotating 45° to a new coordinate system $\Delta R_o' \Delta R_j'$ as shown and using (2) yields

$$1 = \frac{\Delta R_o'^2}{a_o^2} + \frac{\Delta R_j'^2}{a_j^2} \tag{3}$$

Note that this is the equation of a ellipse with semi major and semi minor axes as given by

$$\begin{aligned} a_o &= D_T / \sqrt{1 - \cos \Delta\theta} \\ a_j &= D_T / \sqrt{1 + \cos \Delta\theta} \end{aligned} \tag{4}$$

Expressing the offset point (1) in this rotated frame yields

$$\begin{aligned} R_{so}' &= \frac{D_o}{\sqrt{2} \sin \Delta\theta} (\sin \theta_o + \sin \theta_j) \\ R_{sj}' &= \frac{D_o}{\sqrt{2} \sin \Delta\theta} (\sin \theta_o - \sin \theta_j) \end{aligned} \tag{5}$$

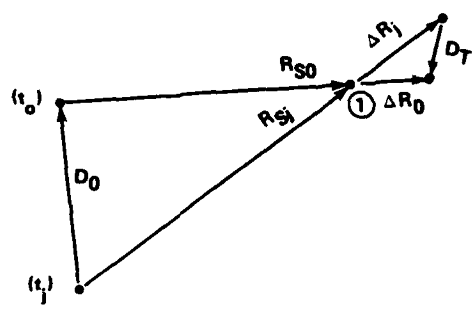


Figure B2. Geometry

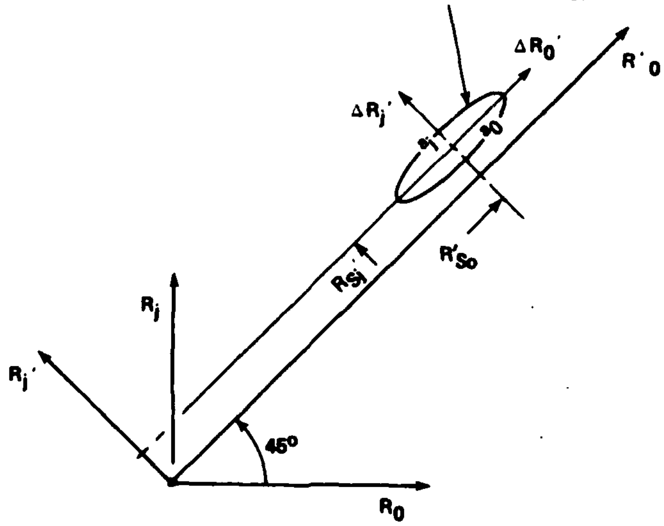


Figure B3. Target Speed Contour

B3 — $\dot{\theta}$ $\dot{f}$ CONTOURS ON A R_oR_j PLOT

$\dot{\theta}$ Contours

Rearrangement of B1(3) yields the R_oR_j contours for $\dot{\theta}_o$ as given by

$$R_j = \left[\frac{\dot{\theta}_o \, \Delta t}{\sin \Delta\theta} \right] R_o \tag{1}$$

Similarily for $\dot{\theta}_j$

$$R_j = \left[\frac{1}{\dfrac{\dot{\theta}_j \, \Delta t}{\sin \Delta\theta}} \right] R_o \tag{2}$$

Thus the $\dot{\theta}_o\dot{\theta}_j$ contours are simply straight lines as shown in Figure B4(a).

$\dot{f}$ Contours

Rearrangement of B1 (3) yields

$$R_o = - \left[\left(\frac{\sin \Delta\theta}{\Delta t} \right)^2 / \lambda \dot{f}_o \right] R_j^2 \tag{3}$$

Similarly for $\dot{f}_j$

$$R_j = - \left[\left(\frac{\sin \Delta\theta}{\Delta t} \right)^2 / \lambda \dot{f}_j \right] R_o^2 \tag{4}$$

These $\dot{f}$ contours (3) and (4) are illustrated in Figure B4(b) including a perturbation on $\dot{f}_o$ and $\dot{f}_j$ to illustrate the effect of $\dot{f}$ changes/errors.

B4 TYPICAL GEOMETRY

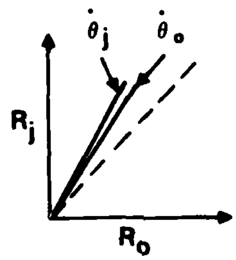
Consider the following conditions: R_o = 60 nmi, R_j = 90 nmi, θ_o = 45°, V_o = 20 kts, and V = 15 kts. Headings are 0° and 180° for own ship and target respectively

Using the law of sines [$60/\sin\phi = 90/\sin 135°$] then $\phi = 28.1°$, thus $\Delta\theta = 16.9°$.

Using the law of cosines, relative motion is obtained as given by

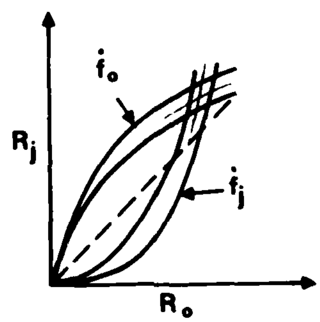
$$D \quad = [60^2 + 90^2 - 2\cdot 60\cdot 90 \cos 16.9°]^{1/2} = 37 \text{ nmi}$$

then the time difference is given by $\Delta t = D/(V_o + V) = 1.03$ hr.



(a)

$\dot\theta$ CONTOURS



(b)

$\dot f$ CONTOURS

81-588

Figure B4. $\dot\theta$ $\dot f$ Contours

APPENDIX C — MEASUREMENT LIKELIHOOD FUNCTION DEVELOPMENT — (F_y)

C1 — F_y FORMULATION

The measurement likelihood function with $\theta_o \theta_j$ integrated out is (refer to A3(6))

$$F_y = \int\int_{\theta_o \theta_j} P(Y_{oj} \mid D_{oj}\, X_G)\, d\theta_o \theta_j \tag{1}$$

Assuming normally distributed measurement errors and $\theta_o \theta_j$ replaced by ΔX_o(for generality) then

$$F_y = K_y \int_{\Delta X_o} e^{-\frac{1}{2}Q}\, d\Delta X_o \tag{2}$$

where $|W_y|$ is the determinant of the inverse of the measurement covariance matrix (diagonal) and

$$Q = (\Delta\tilde{Y}_o - H\,\Delta X_o)^T\ (\Delta\tilde{Y}_o - H\Delta X_o) \tag{3a}$$

$$H = \partial Y/\partial X \tag{3b}$$

$$N_y = \text{number of measurements (elements in Y vector)} \tag{3c}$$

and,

$$\Delta\tilde{Y}_o = \tilde{Y} - Y_o(X_o), \qquad \Delta X_o = X - X_o \tag{3d}$$

$$K_y = (2\pi)^{-N_y/2}(\sqrt{|W_y|}) \tag{3e}$$

Note that the measurements (Y) and the partials (H) have been normalized with respect to their standard deviations (σ). Define $\Delta X'$ to be the change in X relative to the optimum but unknown solution (X^*). Thus,

$$\Delta X_o = \Delta X_o^* + \Delta X' \tag{4}$$

The optimum/mode is given by

$$\Delta X_o{*} = \underbrace{[(H^T H)^{-1} H^T]}_{\lambda} \Delta \tilde{Y}_o \tag{5}$$

The transformation matrix [ ] is designated λ as indicated. Substituting (4) and (5) into (3a) and expanding yields

$$\begin{aligned} Q &= \Delta \tilde{Y}_o^T [(I - H\lambda)^T (I - H\lambda)] \Delta \tilde{Y}_o \\ &\quad - 2 \Delta X'^T [H^T (I - H\lambda)] \Delta \tilde{Y}_o \\ &\quad + \Delta X'^T [H^T H] \Delta X' \end{aligned} \tag{6}$$

Note

(1) $[(I - H\lambda)^T (I - H\lambda)]$ reduces to $[I - H\lambda]$ after noting that λH is an identity matrix and $H\lambda$ is symetric.

(2) $[H^T (I - H\lambda)]$ is zero since expansion yields $[H^T - (H^T H) (H^T H)^{-1} H^T]$ and $(\ )(\ )^{-1} = I$.

With these substitutions (6) reduces to

$$Q = \underbrace{\Delta \tilde{Y}_o^T [I - H\lambda] \Delta \tilde{Y}_o}_{Q_y} + \underbrace{\Delta X'^T [H^T H] \Delta X'}_{Q_x'} \tag{7}$$

Substitution into (2) noteing that Q_y is not a function of X'allows removal from the integral hence

$$F_y = K_y e^{-\frac{1}{2} Q_y} \int_{\Delta X'} e^{-\frac{1}{2} Q_x'} dx' \tag{8}$$

Multiply and divide by the factor $k = [(2\pi^{N_x/2} \sqrt{|H^T H|}]$ to make the intergrals Gaussian. Since the Gaussian integral is unity, then

$$F_y = K_{y1} e^{-\frac{1}{2} Q_y} \tag{9}$$

where $K_{y1} = K_y/k$ and N_x is the number of variables in X' (number integrated out).

Using the definitions of Q_y and λ in (7) and (5) yields

$$Q_y = \Delta\tilde{Y}_o^T [I - H(H^TH)^{-1}H^T] \Delta\tilde{Y}_o \qquad (10)$$

Recall that $\Delta\tilde{Y}_o$ and H are measurements and partials that have been normalized with respect to σ_y. Define the inverse covariance matrix of Y to be W_y. This is a diagonal matrix with elements σ_y^{-2}. Define the matrix $\sqrt{W_y}$ to be the matrix obtained by taking the $\sqrt{\ }$ of each element. Using the (A) subscript to indicate unnormalized (actual) measurements and/or partials yields

$$\Delta Y_o = \sqrt{W_y}\,\Delta Y_A$$

$$H = \sqrt{W_y}\,H_A$$

so that

$$Q = \Delta Y_A^T [W_y - W_yH(H^T W_yH)^{-1} H^TW_y] \Delta Y_A \qquad (11)$$

Partition ΔY_A into the variables integrated out (Y_o) and those remaining (Y_1). W_o and W_1 are the corresponding inverse covariance matrices. Thus

$$\Delta Y = \begin{bmatrix} \Delta Y_1 \\ \Delta Y_o \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ I \end{bmatrix} \qquad (12)$$

Substitution into (9) and (11) and matrix manipulation yields Q_y and W as given by

$$Q_y = \Delta Y_1^T \underbrace{[W_1 - (W_1H_1)(W_o + H_1^TW_1H_1)^{-1}(W_1H_1)^T]}_{W} \Delta Y_1 \qquad (13)$$

K_y Factor

The K_y factor (9) could be determined by matrix algebra, however a more straightforward approach is as follows. Note that the density function integrand of (1) is assumed normal so that F_y is also normal. From (9) and (13)

$$F_y = K_{y1} e^{-\frac{1}{2} \Delta Y_1^T W \Delta Y_1} \qquad (14)$$

In general, the n dimentional normal distribution of x with inverse variance W_x is given by

$$f(x) = \sqrt{\frac{|W_x|}{(2\pi)^n}} e^{-\frac{1}{2} x^T W_x x} \qquad (15)$$

By comparison of these two normal distributions then,

$$K_{y1} = \sqrt{\frac{|W|}{(2\pi)^{N_1}}} \qquad (16)$$

where N_1 is the number of measurables in Y_1 as given by $(N_y - N_x)$.

Final Solution

Substitution of (16) and (13) into (9) then yields

$$F_y = \sqrt{\frac{|W|}{(2\pi)^{N_1}}} \; e^{-\frac{1}{2} \Delta Y_1^T W \Delta Y_1} \qquad (17)$$

where,

$$W = W_1 - (W_1 H_1)(W_o + H_1^T W_1 H_1)^{-1} (W_1 H_1)^T \qquad (18)$$

C2 - APPROXIMATE SOLUTION

Consider the use of 1/o subscript notation to indicate effect/cause respectively. Thus W_1 as currently written would be replaced by $W_{1/1}$ (W with measurement of type 1 was caused by measurement type 1 (same)). $W_{1/o}$ would then mean W with measurements of type 1 caused by measurements of type o. Thus [ ] of (13) is rewritten as

$$W \triangleq [\;] = W_{1/1} - \underbrace{W_{1/1} [H_{1/o} (H_{1/o}^T W_{1/1} H_{1/o} + W_{o/o})^{-1} H_{1/o}^T] W_{1/1}}_{[\;]_1} \qquad (19)$$

A covariance matrix will be designated by M, the inverse of W (i.e., $M_{1/1} = W_{1/1}^{-1}$). It can be shown that a covariance matrix of one variable type can be transformed to another by the matrix operation HMH^T. For example, transforming from type o to type 1, yields

$$M_{1/o} = H_{1/o} M_{o/o} H_{1/o}^T \tag{20}$$

Similarly the inverse covariance matrices can be transformed using the operation $H^T WH$. Thus

$$W_{o/1} = H_{1/o}^T W_{1/1} H_{1/o} \tag{21}$$

then the $[\ ]_1$ term of (19) can be dissected as follows

$$[\ ]_1 = \underbrace{H_{1/o} \underbrace{\underbrace{(\underbrace{H_{1/o}^T W_{1/1} H_{1/o}}_{W_{o/1}} + W_{o/o})}_{W_{o/o,1}}{}^{-1}}_{M_{o/o,1}} H_{1/o}^T}_{M_{1/o,1}} \tag{22}$$

Thus $[\ ]_1 = M_{1/o,1}$ is the joint covariance matrix of type 1 variables after estimation/smoothing using both type 1 and o variables. It is obtained by determining the smoothed covariance matrix in terms of type o variables ($M_{o/o,1}$) and then transforming to type 1 variables.

An alternate approach to obtaining $M_{1/o,1}$ and easier to interpret and simplify, is to first transform $W_{o/o}$ to $W_{1/o}$ and then combine/smooth in type 1 variables so that

$$[\ ]_1 = M_{1/o,1} = [\, \underbrace{\underbrace{(H_{1/o} \underbrace{W_{o/o}^{-1}}_{M_{o/o}} H_{1/o}^T)}_{M_{1/o}}{}^{-1}}_{W_{1/o}} + W_{1/1}\,]^{-1} \tag{23}$$

Note that this would simplify if $M_{1/o}$ was assumed to be diagonal, then $W_{1/o}$ and $[\ \ ]_1$ are diagonal and (19) reduces to

$$W^* = W_{1/1} - W_{1/1}\,(W_{1/o} + W_{1/1})^{-1}\,W_{1/1} \tag{24}$$

Since $W_{1/1}$ is diagonal and $W_{1/o}$ assumed diagonal then W^* is also diagonal. Thus any/each diagonal element can be determined independent of the others and inverses become reciprocals. Consider the i th diagonal element and for simplicity let $a = W_{1/1}$ and $b = W_{1/o}$.

Then,

$$W_i^* = a_i - a_i \frac{1}{b_i + a_i}\, a_i \tag{25}$$

which reduces to

$$W_i^* = \frac{1}{1/a_i + 1/b_i} \tag{26}$$

where it is noted that $1/a_i$ and $1/b_i$ are the covariances $M_{1/1i}$ and $M_{1/oi}$ which are simply $\sigma_{1/1i}^2$ and $\sigma_{1/oi}^2$. Note $\sigma_{1/1i}^2$ is the variance of a type 1 measurement (Y_1). $\sigma_{1/o}^2$ is the variance of the errors in predicting Y_1 due to the type o errors (Y_o).

$M_{1/o\ i}$ is obtained using the transformation of (20) so that

$$W_i^* = \frac{1}{M_{1/1\ i} + H_{1/o\ i}\, M_{o/o}\, H_{1/o\ i}^T} \tag{27}$$

For notational convenience this will be written as

$$W_i^* = \frac{1}{M_{1i} + H_{1i}\, M_o\, H_1^T} \tag{28}$$

Recall that M_{1i} is the i th covariance $(\sigma_{\dot{\theta}o}^2, \sigma_{\dot{\theta}j}^2, \ldots)$, $(\theta_o\theta_j$ not included). H_{1i} is the $(N_1 \times 2)$ partial derivitive matrix of the N_1 type 1 variables with respect to the two type o variables $(\theta_o\theta_j)$. M_o is a 2 x 2 diagonal covariance matrix with diagonal elements $\sigma_{\theta o}^2\ \sigma_{\theta j}^2$.

Thus, since W* is diagonal, replacing W by W* in (13) yields the simple expression

$$Q_y^* = \sum_i^{N_1} \Delta Y_{1i} W_i^* \tag{29}$$

Since W* is diagonal it's determinant |W*| is simply the product of the N_1 diagonal elements so that

$$|W^*| = \prod_i^{N_1} W_i^* \tag{30}$$

$$K_{y1}^* = \sqrt{\frac{\prod^{N_1} W_i^*}{(2\pi)^{N_1}}} = \prod^{N_1} \sqrt{\frac{W_i^*}{2\pi}} \tag{31}$$

Substitution of (31) and (29) into (9) and adding the (*) notation for the approximate solution yields

$$F_y^* = \left[\prod_i^{N_1} \sqrt{\frac{W_i^*}{2\pi}}\right] e^{-\frac{1}{2}\sum_i^{N_1} (\Delta Y_{1_i}{}^2 W_i^*)} = \prod_i^{N_1} \left[\sqrt{\frac{W_i^*}{2\pi}}\; e^{-\frac{1}{2}\Delta Y_{1i}{}^2 W_i^*}\right] \tag{32}$$

where,

$$W_i^* = \frac{1}{M_{1i} + H_{1i} M_o H_{1i}{}^T} \tag{33}$$

Note that if the Y_1 errors (M_{1i}) are large relative to the M_o variance contributions ($H_{1_i} M_o H_{1_i}{}^T$) then $W_i^* \to 1/M_{1i}$.

C3—AN EXAMPLE

Consider the typical condition with bearing, bearing rate and frequency rate measurements ($\theta_o \dot{\theta}_o \dot{f}_o$ and $\theta_j \dot{\theta}_j \dot{f}_j$) for segments oj. Then, assuming linear motion (refer to B1)

$$\Delta Y_1 = [\tilde{Y}_1 - Y_1(X_G)] = \begin{bmatrix} \tilde{\dot{\theta}}_o - r\dot{\phi} \\ \tilde{\dot{\theta}}_j - r^{-1}\dot{\phi} \\ \lambda \tilde{\dot{f}}_o + r R_j \dot{\phi}^2 \\ \lambda \tilde{\dot{f}}_j + r^{-1} R_o \dot{\phi}^2 \end{bmatrix} \tag{1}$$

$$H_1 = \begin{bmatrix} -r\dot{\phi}' & , & - \\ -r^{-1}\dot{\phi}' & , & - \\ +2r\,R_j\dot{\phi}\dot{\phi}' & , & - \\ +2r^{-1}\,R_o\dot{\phi}\dot{\phi}', & & - \end{bmatrix} \triangleq \begin{bmatrix} H_{11} \\ H_{12} \\ H_{13} \\ H_{14} \end{bmatrix} = \frac{\partial\,\dot{\theta}_o\dot{\theta}_j\dot{f}_o\dot{f}_j}{\partial\,\theta_o\theta_j} \qquad (2)$$

where λ is wavelength, $\lambda\dot{f}$ is considered the measurable rather than $\dot{f}$ and

$$r \triangleq \frac{R_j}{R_o}, \ \dot{\phi} \triangleq \frac{\sin\Delta\theta}{\Delta t}, \ \dot{\phi}' \triangleq \frac{\cos\Delta\theta}{\Delta t} \qquad (3)$$

$\Delta\theta$ and Δt are the measured bearing and time differences. The second column of H_1 is the negative of the first column.

W_1 is 4 x 4 ($N_1 = 4$), diagonal with elements that are the reciprocal of the measurement covariances (M_1) elements ($\sigma_{\dot{\theta}_o}{}^2\ \sigma_{\dot{\theta}_j}{}^2\ \sigma_{\lambda\dot{f}_o}{}^2\ \sigma_{\lambda\dot{f}_j}{}^2$). ( ) are the elements of M_{1i}, i = 1,2,3,4. W_o (similarly) is 2 x 2 diagonal, with elements that are the reciprocal of the $\theta_o\theta_j$ covariances (M_o) elements. If the simplified form of W_i* is used then the M_i* covariances are

i	$M_i^* \triangleq 1/W_i^*$
1	$\sigma_{\dot{\theta}_o}{}^2 + (r\,\dot{\phi}')^2\,\sigma_o{}^2$
2	$\sigma_{\dot{\theta}_j}{}^2 + (r^{-1}\,\dot{\phi}')^2\,\sigma_o{}^2$
3	$\sigma_{\lambda\dot{f}_o}{}^2 + (2r\,R_j\dot{\phi}\dot{\phi}')^2\,\sigma_o{}^2$
4	$\sigma_{\lambda\dot{f}_j}{}^2 + (2r^{-1}\,R_o\dot{\phi}\dot{\phi}')^2\,\sigma_o{}^2$

(4)

where,

$$\sigma_o{}^2 = \sigma_{\theta_o}{}^2 + \sigma_{\theta_j}{}^2$$

then, for example, one of the F_y factors(F_{y1})(applying to the $\dot{\theta}_o$ measurement) is given by

$$F_{y1} = \frac{1}{\sqrt{2\pi\,M_1^*}}\ \exp\left[-\tfrac{1}{2}\left(\tilde{\dot{\theta}}_o - \frac{R_j \sin\Delta\theta}{R_o\,\Delta t}\right)^2 \frac{1}{M_1^*}\right] \qquad (5)$$

C4 – REDUCTION USING $\Delta\theta$ FOR SINGLE SHIP WITH LINEAR MOTION

As shown in B1(3) the measurables $\dot{\theta}_o \dot{f}_o \dot{\theta}_j \dot{f}_j$ (plus $\theta_o \theta_j$) are a function of $\Delta\theta$ (and $R_o R_j$) and hence the double integral of C1 (1) reduces to a single integral with variate $\Delta\theta$ having variance $\sigma_{\Delta\theta}^2 = \sigma_{\theta_o}^2 + \sigma_{\theta_j}^2$. Thus the term $(\ )^{-1}$ of C1(13) is a scalar given by (note $H_i \stackrel{\Delta}{=} \partial Y_{1i}/\partial\Delta\theta$)

$$M_{o/o,1} = 1/\left[\frac{1}{\sigma_{\Delta\theta}^2} + \sum_i^{N_1}\left(H_i \frac{1}{\sigma_{Y_{1i}}}\right)^2\right] \tag{1}$$

Thus manipulation of C1(13) yields

$$W = W_1 - (W_1 H_1)(W_1 H_1)^T M_{o/o,1}$$

and

$$Q = \Delta Y_1^T W_1 \Delta Y_1 - (H_1^T W_1 \Delta Y_1)^2 M_{o/o,1} \tag{2}$$

or

$$= \Sigma\left(\frac{\Delta Y_{1i}}{\sigma_{Y_{1i}}}\right)^2 - \left[\Sigma H_i \frac{\Delta Y_{1i}}{\sigma_{Y_{1i}}^2}\right]^2 M_{o/o,1} \tag{3}$$

APPENDIX D — DEVELOPMENT OF FREQUENCY SPECTRUM LIKELIHOOD RATIO

D1—LIKELIHOOD RATIO

Assuming cell to cell independence and n_c cells then

$$\Lambda_S = \prod_{i}^{n_c} \Lambda_i \tag{1}$$

where the likelihood ratio for a typical ith cell is given by

$$\Lambda_i = \frac{P(S_i D_i | A)}{P(S_i D_i | \overline{A})} \tag{2}$$

$S_i D_i$ is the amplitude/detection information for the ith frequency cell. A is the condition that the two data segments (complete frequency spectra) are associated; i.e., both have a common origin (target). $\overline{A}$ means they are not associated. These can be interpreted as 0-1 hypothesis ($H_1 = A$, $H_o = \overline{A}$).

D2—MEASUREMENT AND STATES DEFINED

The target will be modeled as a source with n_T frequency lines. This is the mean number of lines detected in two segments (segment o,j). The line amplitude (X_i) (without fluctuation) is defined by the probability density function $P(X_i)$ or simply $P(X)$. Fluctuation is considered log normal with standard deviation σ_F^2. Probability of fade is P_F. Extraneous lines are also present; uncorrelated from segment to segment with the mean number given by n_E.

Thus the measurements are

$$S = \begin{bmatrix} S_o \\ S_j \end{bmatrix} = \begin{bmatrix} S_{o1} \ldots & S_{oi} \ldots & S_{on_c} \\ S_{j1} \ldots & S_{ji} \ldots & S_{jn_c} \end{bmatrix} \begin{array}{l} \text{- segment o} \\ \text{- segment j} \end{array}$$

and the target variables are (n_T, σ_F, P_F) and amplitude S_x with Pdf $P(S_x)$.

The prior conditions are defined in terms of detection (D) and correlation (C) states for each association condition (A or $\overline{A}$).

There are four binary detection states D_{11}, D_{01}, D_{10}, D_{00} where D_{11} means the target (T) is detected in both segments, D_{01} means no detection in one segment and detection in the other, etc.

There are two correlation states. One where no extraneous (E) lines are present ($\underline{C}$) in the cell; the other where E lines do exist (uncorrelated) ($\overline{C}$) in the cell (in one or both segments). Evaluation of (2) for each cell depends on the likelihood functions of the measurements ($\tilde{S}$, $\tilde{D}$) and the states [S, D, C]. Assuming the amplitude distributions of target and extraneous lines are the same, then (Appendix E1) the expected likelihood ratio of (2) for one of the cells of type m (defined below) is given by

$$\Lambda_m = \underbrace{\left[\frac{P(S|\underline{C}D)}{P(S|\overline{C}D)}\right]}_{\Lambda_{sm}} \underbrace{\frac{P(D\underline{C}|A)}{P(D\overline{C}|\overline{A})}}_{\lambda_{m1}} + \underbrace{\frac{P(D\overline{C}|A)}{P(D\overline{C}|\overline{A})}}_{\lambda_{m2}} \tag{5}$$

where Λ_{sm} is the expected likelihood ratio conditioned on the CD states of a cell. The other terms are a priori probability of the cell DC states conditional on association/nonassociation (the existance/nonexistance of the same target source in both segments).

The form of the terms of (5) depend on whether the cell contains two lines (detection in both cells), one line or no lines. Using index m to make this distinction, let m = 2, for two lines, m = 1, for 1 line and m = 0 for no lines. Then (5) is written as (for the ith cell)

$$\Lambda_m = \Lambda_{sm} \lambda_{m1} + \lambda_{m2} \qquad m = 0,1,2 \tag{6}$$

Then the total likelihood ratio (1) is given by

$$\Lambda_S = \prod^{n_2} (\Lambda_{s2} \lambda_{21} + \lambda_{22}) \prod^{n_1} (\Lambda_{s1} \lambda_{11} + \lambda_{12}) \prod^{n_0} \lambda_o \tag{7}$$

where,

n_2 = number of cells with two lines (m = 2)

n_1 = number of cells with one line (m = 1)

n_0 = number of cells with no lines (m = 0)

D3 LIKELIHOOD RATIO (Λ_{sm}) DEVELOPMENT

Λ_{sm} is evaluated by integrating over X and applying Bayes' Rule (Appendix E2) to obtain

$$\Lambda_{sm} = \underbrace{\frac{\int_X P(S_o D_o|X)\, P(S_j D_j|X)\, P(X)}{[\int_{X_o} P(S_o D_o|X_o)\, P(X_o)]\; [\int_{X_j} P(S_j D_j|X_j)\, P(X_j)\,]}}_{F_m} \div \underbrace{\frac{P(D|C)}{P(D|\overline{C})}}_{G_m} \qquad (8)$$

If S_o is not detected then $P(S_o\overline{D}_o|X) = P(\overline{D}_o|X)$; similarly for S_j (Appendix E2). Note for cells with no lines ($\overline{D}_o$, $\overline{D}_j$), (8) reduces to [$\Lambda_{sm} = 1$].

If only the detect/no detect event is recorded (amplitude not recorded) as in a clipped processor, then (Appendix E2) [$\Lambda_{s1} = \Lambda_{s2} = 1$]. Consider each in turn.

Λ_{s2} — Double Detection

For detection state D_{11} where detection occurs in both the o and j segment (m = 2) then,

$$\Lambda_{s2} = \underbrace{\frac{\int N_o N_j\, P(X)}{\int N_o\, P(X) \int N_j\, P(X)}}_{F_2} \div \underbrace{\frac{P(D_o\, D_j)}{P(D_o)\, P(D_j)}}_{G_2}$$

where N_o and N_j are normal distributions of the measured amplitude conditioned on amplitude state X. These are likelihood functions; a function of X. D_o and D_j are the detection events of segments o,j. $P(D_o, D_j)$ for instance is the joint probability of the measured detection events in cells o, j given that a detectable target line exists.

As developed in Appendix E5

$$\Lambda_{s2} = N\left(\frac{S_o - S_j}{\sqrt{2}\,\sigma_F}\right) \frac{e^{\lambda\,(S_{ave}\; - T + .5\,\sigma_F)}}{\lambda} \qquad (9)$$

where S_o and S_j are the measured amplitudes in segments o and j, S_{ave} is the average amplitude, T is the threshold, $\sigma_F{}^2$ is the variance of the log normal fluctuation distribution and λ is a factor in the prior amplitude distribution (E5). It can be shown that $P_F = \lambda\sigma$ where P_F is probability of fade.

Λ_{s1} — Single Detections

For cells with a detection in one segment (e.g., o) but not the other (j) then;

$$\Lambda_{s1} = \underbrace{\frac{\int N_o P(\overline{D}_j|X)\,P(X)}{\int N_o P(X) \cancel{\int P(\overline{D}_j|X)P(X)}}}_{F_1} \div \underbrace{\frac{P(D_o\overline{D}_j)}{P(D_o)\,\cancel{P(\overline{D}_j)}}}_{G_1} \tag{10}$$

where $\overline{D}$ is the measured event that no detection occurs in the jth segment. The second intergral in the denominator is, by inspection, equal to P $(\overline{D}_j)$ and hence cancels as indicated. Following the same assumptions and techniques as developed in Appendix E6.

$$\Lambda_{s1} = \frac{1}{2\lambda\sigma}\, e^{-.8\left(\frac{S_o - T}{\sqrt{2}\,\sigma_F}\right)} \tag{11}$$

Note that the same expression applies when only the jth segment is detected; simply replace S_o with S_j.

Λ_o — No Detection

By inspection (Appendix E2)

$$\Lambda_o = 1$$

This completes the development of the likelihood (Λ_{sm}) terms of (7). Now consider the a priori terms (λ_{m1}, λ_{m2}) for m = 1,2.

D4 — PRIAR DISTRIBUTION TERMS OF D2(6) DEVELOPED

These terms (λ_{m1}, λ_{m2}) express the a priori probability ratios for the various detection/correlation states (DC) conditioned on the presence (A) or non presence ($\overline{A}$) of a common target in both segments.

As developed in Appendix E7, these ratios are given in terms of the target/extraneous line ("noise") parameters previously defined (n_T, n_E, P_F). The results are given in Table D-1.

TABLE D-1 — λ TERMS

λ_2 — Double Detection

$$\lambda_{21} = (1 - P_F)\, n_T n_C \,/\, n_E'^2$$

$$\lambda_{22} = n_E\,(n_E + n_T) \,/\, n_E'^2$$

λ_1 — Single Detection

$$\lambda_{11} = \frac{P_F}{2}\, n_T \,/\, n_E'$$

$$\lambda_{12} = n_E \,/\, n_E'$$

where,

$$n_E' = n_E + (1 - \frac{P_F}{2})\, n_T$$

No Detection

$$\prod^{n_o} \lambda_o = e^{n_T (1 - P_F)} \qquad \text{(Refer to E8)}$$

where,

$$n_o = n_C - (n_1 + n_2)$$

Thus the complete spectrum likelihood ratio is given by (7) where the terms Λ_{s2} and Λ_{s1} are given by (9) and (11) and the λ terms given in Table D-1.

APPENDIX E — BACKUP ANALYSIS FOR APPENDIX D(Λ_S DEVELOPMENT)

E1 — EXPANSION/DEVELOPMENT OF $P(SD|A)/P(SD|\overline{A})$

For a given cell, Λ_s is given by (cell subscript (i) is implied).

$$\Lambda_S = \frac{P(\tilde{S}\tilde{D}|A)}{P(\tilde{S}\tilde{D}|\overline{A})} \tag{1}$$

where $\tilde{S}$, $\tilde{D}$ are measured amplitude and detection events respectively, and A, $\overline{A}$ are the association, non association conditions respectively for the complete spectrum. Recall that D can take on one of four "values"; detected in both segments D_{11}; detected in only one (D_{01}, D_{10}) or no detections (D_{oo}).

S D is a function of whether or not the cell contains an extraneous line. If it does then the segments are uncorrelated ($\overline{C}$) otherwise they are correlated ($\underline{C}$).

Including the correlation state variable (C) with the two possible values ($\underline{C}$, $\overline{C}$), and summing over the two values yields

$$\Lambda_S = \sum_c P\,(SDC \mid A) \,/\, \sum_c P\,(SDC \mid \overline{A}) \tag{2}$$

Applying Bayes' Rule yields

$$\Lambda_S = \sum_c P(S|CDA)\,P(DC|A) \,/\, \sum_c P(S|CD\overline{A})\,P(DC|\overline{A})$$

$$= \frac{P(S|\underline{C}DA)\,P(D\underline{C}|A) + P(S|\overline{C}DA)\,P(D\overline{C}|A)}{P(S|\underline{C}D\overline{A})\,P(D\underline{C}|\overline{A}) + P(S|\overline{C}D\overline{A})\,P(D\overline{C}|\overline{A})} \tag{3}$$

Note that the cells cannot be correlated if unassociated hence $P(D\underline{C}|\overline{A}) = 0$; hence the first denominator term vanishs.

Assuming that target and extraneous lines have the same distributions, than the amplitude densities (S| –) are a function only of the correlation condition; the association condition ($A, \overline{A}$) being irrelevent so that $P(S|\underline{C}DA) = P(S|\underline{C}D)$, thus

$$\Lambda_S = \underbrace{\left[\frac{P(S|\underline{C}D)}{P(S|\overline{\underline{C}}D)} \right]}_{\Lambda_{sm}} \cdot \frac{P(D\underline{C}|A)}{P(D\overline{C}|\overline{A})} + \frac{P(D\overline{C}|A)}{P(D\overline{C}|\overline{A})} \tag{4}$$

E2 – EVALUATION OF Λ_{sm}

$$\Lambda_{sm} \overset{\Delta}{=} \frac{P(S|\underline{C}D)}{P(S|\overline{\underline{C}}D)} \tag{1}$$

$$= \frac{P(SD|\underline{C}) \,/\, P(D|\underline{C})}{P(SD|\overline{C}) \,/\, P(D|\overline{C})} \tag{2}$$

The amplitude term of the numerator and denominator are determined by introducing the state variable X to the joint density function and then integrating it out to obtain the desired marginal densities. Thus,

$$\begin{aligned} P(SD|C) &= \int_X P(SD|XC)\, P(X|C)\, dx \\ P(SD|\overline{C}) &= \int_X P(SD|X\overline{C})\, P(X|\overline{C})\, dx \end{aligned} \tag{3}$$

Substitution into (2) then yields

$$\Lambda_{sm} = \frac{\int_X P(SD|XC)\, P(X|C) dx}{\int_X P(SD|X\overline{C})\, P(X|\overline{C}) dx} \div \frac{P(D|C)}{P(D|\overline{C})} \tag{4}$$

Recall $S = (S_o S_j)$. For the correlated condition ($\underline{C}$) a single state (X) applies. For $\overline{C}$, different X apply $(X_o X_j)$ so that (dropping the correlation notations)

$$\Lambda_{sm} = \frac{\int_X P(S_o D_o|X)\, P(S_j D_j|X)\, P(X)\, dx}{\int_{X_o} P(S_o D_o|X_o)\, P(X_o)\, dx_o \int_{X_j} P(S_j D_j|X_j)\, P(X_j)\, dx_j} \div \frac{P(D|C)}{P(D|\overline{C})} \tag{5}$$

Consider the detection and no detection condition that may be measured for the o or j segment. Applying Bayes' Rule for segment o (similarly for j) yields the alternate expressions

$$P(S_o D_o|X) = P(D_o|S_o X)\, P(S_o|X) \tag{6}$$

$$= P(D_o|X)\, P(S_o|D_o X) \tag{7}$$

If detection occurs ($D_o = \underline{D}_o$) then applying (6)

$$P(D_o|S_oX) = P(D_o|S_o) = 1$$

thus, (8)

$$P(S_o\underline{D}_o|X) = P(S_o|X)$$

If no detection occurs ($D_o = \overline{D}_o$) then applying (7)

$$P(S_o|\overline{D}_oX) = P(S_o|\overline{D}_o) = 1$$

thus, (9)

$$P(S_o\overline{D}_o|X) = P(\overline{D}_o|X)$$

In conclusion, the result (Λ_{sm}) is given by (5) noteing per (8), (9) that

$$\begin{aligned} P(S_oD_o|X) &= P(S_o|X) \quad \text{for the detection condition} \\ &= P(\overline{D}_o|X) \quad \text{for the no detection condition} \end{aligned} \tag{10}$$

The same applies for segment j.

<u>Clipped</u>

If only the detection event is recorded (no amplitudes) then

$$P(S_oD_o|X) = P(D_o|X)$$

or, (11)

$$P(S_jD_j|X) = P(D_j|X)$$

Then the numerator integral of (5) equals $P(D_oD_j|C)$ or simply $P(D|C)$ and the denominator integral equals $P(D_o)$ $P(D_j)$ or simply $P(D|\overline{C})$. These terms then cancel with corresponding terms so that

$$\Lambda_{sm} = 1 \tag{12}$$

E3 DETECTION TERM DEVELOPMENT WITH Q APPROXIMATIONS

Several detection terms will be developed as given by

1. $P(D_o)^{(1)}$ = Probability of detection given the existance of a detectable line in a single segment cell.

2. $P(D_o D_j) \triangleq P(D_o D_j | C)$

   = Joint probability of detection given the existance of a correlated line (same target) in a common cell of both segments.

3. $P(D_o \overline{D}_j) = P(D_o \overline{D}_j | C)$

   = Joint probability of detection in one cell and no detection ($\overline{D}$) in the other given the same conditions as in 2.

P(D) is obtained by introducing the state variable (X) and integrating it out so that

$$P(D_o) = \int_X P(D_o X)\, dx = \int_X P(D_o|X)\, P(X)\, dx \tag{1}$$

Similarly,

$$P(D_o D_j) = \int_X P(D_o|X)\, P(D_j|X)\, P(X)\, dx \tag{2}$$

$$P(D_o \overline{D}_j) = \int_X P(D_o|X)\, P(\overline{D}_j|X)\, P(X)\, dx \tag{3}$$

$P(D_o|X)$ and $P(\overline{D}_o|X)$ Development

$P(D_o|X)$ is the probability that the amplitude (S_o) exceeds a threshold (T), conditioned on X. The conditional probability density function of amplitude is $P(S|X)$ so that

$$P(D_o|X) = \int_{S=T}^{\infty} P(S|X)\, ds \tag{4}$$

Assuming a Gaussian fading model with variance σ^2 as previously discussed, then

$$P(S|X) = \frac{e^{-\frac{1}{2}\left(\frac{S-X}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma} \triangleq N\left(\frac{S-X}{\sigma}\right) \tag{5}$$

(1) $P(D_j) = P(D_o)$ and $P(\overline{D}_o D_j) = P(D_o \overline{D}_j)$

Then, substitution into (4) and defining $P(D|X) = Q$ or $Q(X)$ yields

$$Q \quad \triangleq P(D_o|X) = \int_{S=T}^{\infty} N\left(\frac{S-X}{\sigma}\right) ds \tag{6}$$

Note that this has the form of an error function (ERF). For $X>T$, a good approximation is

$$\begin{aligned} Q &= P(D_o|X) \cong \left[1 - .5\, e^{-.8\left(\frac{X-T}{\sigma}\right)}\right] \\ &= Q\left(\frac{X-T}{\sigma}\right) \end{aligned} \tag{7}$$

Since $P(D_o|X) = P(D_j|X)$, then $P(D_o|X) \cdot P(D_j|X) = Q^2$. An approximation to Q^2 is obtained by inspection of plots of Q and Q^2 (Refer to Figure E1). Q^2 is essentially the same as Q with an X shift of about 0.5σ.

Thus,

$$Q^2 \quad \cong Q\left(\frac{X - T - .5\sigma}{\sigma}\right) \tag{8}$$

Thus, from (7)

$$Q^2 \quad \triangleq P(DD|X) \cong 1 - .5\, e^{-.8\left(\frac{X-T-.5\sigma}{\sigma}\right)} \tag{9}$$

Note that if a line is present (detectable) it is either detected or not detected so that

$$\begin{aligned} P(\overline{D}|X) &= 1 - P(D|X) \\ &= 1 - Q \end{aligned} \tag{10}$$

Substitution of (7), (9) and (10) into (1) – (3) yields

$$\begin{aligned} P(D_o) = P(D_j) &= \int_X Q\, P(X)\, dx \\ P(D_o D_j) &= \int_X Q^2\, P(X)\, dx \\ P(D_o \overline{D}_j) = P(\overline{D}_o D_j) & \\ &= \int_X Q(1-Q)\, P(X)\, dx \end{aligned} \tag{11}$$

where Q and Q^2 are given by (7) and (9).

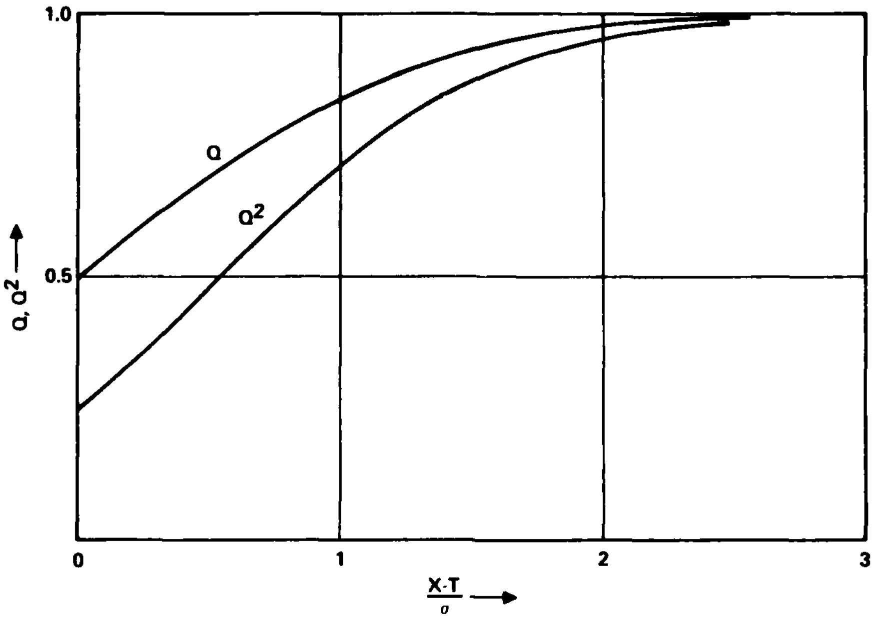


Figure E1 — Q and Q^2

E4 — EVALUATION OF N_Δ

The expression to be developed is given by

$$N_\Delta = \int_X N_o N_j \, dx$$

where N_o, N_j are normal distributions of S_o, S_j with true value X. Thus, for equal variances

$$N_\Delta = \int_X \frac{e^{-\frac{1}{2}\left(\frac{S_o - X}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma} \cdot \frac{e^{-\frac{1}{2}\left(\frac{S_j - X}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma} \, dx \tag{1}$$

Refer to Figure E2.

Completing the square yields

$$N_\Delta = \int_X \frac{e^{-\frac{1}{2}\left[\left(\frac{S_o - S_j}{\sqrt{2}\,\sigma}\right)^2 + \left(\frac{X - \frac{S_o + S_j}{2}}{\sigma/\sqrt{2}}\right)^2\right]}}{\sqrt{2\pi}\,\sqrt{2}\,\sigma \cdot \sqrt{2\pi}\,\sigma/\sqrt{2}} \, dx \tag{2}$$

Since the first exponent is independent of X, it can be removed from the integral so that

$$N_\Delta = \frac{e^{-\frac{1}{2}\left(\frac{S_o - S_j}{\sqrt{2}\,\sigma}\right)^2}}{\sqrt{2\pi}\,\sqrt{2}\,\sigma} \int_X \frac{e^{-\frac{1}{2}\left(\frac{X - \frac{S_o + S_j}{2}}{\sigma/\sqrt{2}}\right)^2}}{\sqrt{2\pi}\,\sigma/\sqrt{2}} \, dx \tag{3}$$

Since the integral is a normal distribution, integration yields unity so that

$$N_\Delta = \frac{e^{-\frac{1}{2}\left(\frac{S_o - S_j}{\sqrt{2}\,\sigma}\right)^2}}{\sqrt{2\pi}\,\sqrt{2}\,\sigma}$$

or,

$$\boxed{N_\Delta = N\left(\frac{S_o - S_j}{\sqrt{2}\,\sigma}\right)} \tag{4}$$

For the more general case where the σ's are unequal, define $a = S_o$, $b = S_j$, then

$$N_\Delta = \int_X \frac{e^{-\frac{1}{2}\left(\frac{a - x}{\sigma_a}\right)^2}}{\sqrt{2\pi}\,\sigma_a} \cdot \frac{e^{-\frac{1}{2}\left(\frac{b - x}{\sigma_b}\right)^2}}{\sqrt{2\pi}\,\sigma_b} \, dx \tag{5}$$

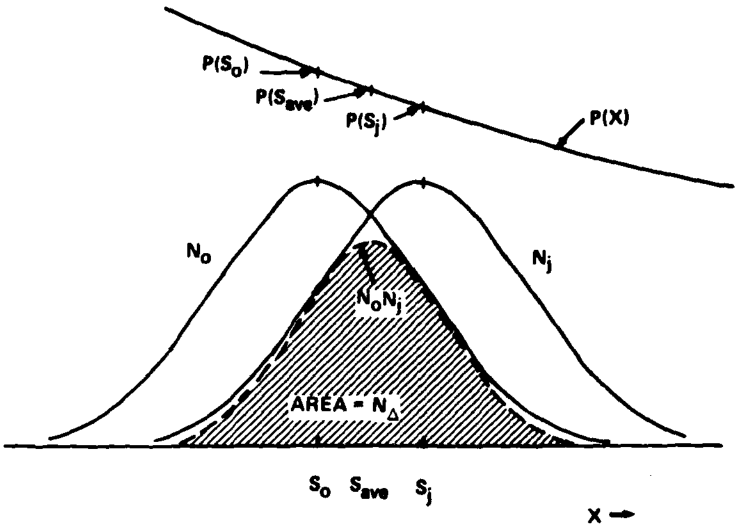


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Figure E2 — N_Δ

completing the squares yields

$$N_\Delta = \left[\frac{\sigma_{ab}\,\sigma_{\hat{x}}}{\sigma_a\,\sigma_b}\right]_1 N\left(\frac{a-b}{\sigma_{ab}}\right) \int_X N\left(\frac{\hat{x}-x}{\sigma_{\hat{x}}}\right) dx \tag{6}$$

where,

$$\sigma_{ab} = \sqrt{\sigma_a^{\,2} + \sigma_b^{\,2}} \tag{7}$$

$$\sigma_{\hat{x}} = \sqrt{\frac{1}{\frac{1}{\sigma_a^{\,2}} + \frac{1}{\sigma_b^{\,2}}}} \tag{8}$$

thus,

$$[\ \ ]_1 = 1$$

The integral is also unity, thus (6) reduces to

$$\boxed{N_\Delta = N\left(\frac{a-b}{\sigma_{ab}}\right)} \tag{9}$$

or

$$N_\Delta = \frac{1}{\sqrt{2\pi}\,\sigma_{ab}}\, e^{-\frac{1}{2}\left(\frac{a-b}{\sigma_{ab}}\right)^2} \tag{10}$$

E5 — Λ_{S2} DEVELOPMENT

$$\Lambda_{S2} \stackrel{\Delta}{=} F_2 / G_2$$

F_2 Development

$$F_2 = \frac{\int_X N_o N_j\, P(X)\, dx}{\int_X N_o\, P(X)\, dx \int_X N_j\, P(X)\, dx} \tag{1}$$

Consider the numerator of (1) and Figure E2.

As shown in the development of N_Δ (Appendix E4), $N_o N_j$ reduces to a single normal distribution with mean at $(S_o + S_j) / 2 \stackrel{\Delta}{=} S_{ave}$.

Assuming P (X) is linear in the region of S_{ave} and noteing that $N_o N_j$ is symmetrical about S_{ave}, then the numerator is

$$F_2 \text{ (num)} = P(S_{ave}) \int_X N_o N_j \, dx \tag{2}$$

The integral term is N_Δ (Appendix E4) so that

$$F_2 \text{ (num)} = N_\Delta P(S_{ave}) \tag{3}$$

Consider the denominator of (1) and Figure E2. Assuming, again, that P (X) is linear in the region $S_o \pm \sigma$ and $S_j \pm \sigma$, then

$$\begin{aligned} F_2 \text{ (den)} &= [\, P(S_o) \int_X N_o \, dx \,] \cdot [\, P(S_j) \int_X N_j \, dx \,] \\ &= P(S_o)\, P(S_j) \end{aligned} \tag{4}$$

Finally, since $F_2 \stackrel{\Delta}{=} F$ (num)/F (den) then

$$F_2 = \frac{N_\Delta \, P(S_{ave})}{P(S_o)\, P(S_j)} \tag{5}$$

Assuming the exponential form for P (X) : $P(X) = \lambda e^{-\lambda X}$ then

$$\begin{aligned} \frac{P(S_{ave})}{P(S_o)\, P(S_j)} &= \frac{\lambda e^{-\lambda S_{ave}}}{\lambda e^{-\lambda S_o} \; \lambda e^{-\lambda S_j}} \\ &= \frac{e^{+\lambda S_{ave}}}{\lambda} \end{aligned} \tag{6}$$

Substitution into (5) yields

$$F_2 = N_\Delta \frac{e^{\lambda S_{ave}}}{\lambda} \tag{7}$$

where

$$N_\Delta = N\left(\frac{S_o - S_j}{\sqrt{2}\,\sigma}\right) \tag{8}$$

<u>G_2 Development</u>

$$G_2 = \frac{P(D_oD_j)}{P(D_o)^2} \tag{9}$$

Using the approximation of P (DD) and P (D) given by (7), (8) and (11) of E3 yields

$$G_2 = \frac{\int_X Q^2 P(X)\,dx}{[\int_X QP(X)\,dx]^2} = \frac{\int_X Q\left(\frac{X - T - .5\sigma}{\sigma}\right) P(X)\,dx}{[\int_X Q\left(\frac{X-T}{\sigma}\right) P(X)\,dx]^2} \tag{10}$$

Assuming that P(X) is broad relative to σ, then Q and Q^2 are essentially step functions at $\left(\frac{X-T}{\sigma}\right)$ = 0 and 0.5 respectively so that

$$G_2 = \frac{\int_{T+0.5\sigma}^{\infty} P(X)\,dx}{[\int_T^{\infty} P(X)\,dx]^2} \tag{11}$$

Assuming an exponential distribution for P(X) as given by

$$P(X) = \lambda e^{-\lambda X} \tag{12}$$

then substitution into (11) and integration yields

$$G_2 = e^{+\lambda\,(T - 0.5\,\sigma)} \tag{13}$$

Combining the F_2 and G_2 factors yields

$$\Lambda_{S2} \triangleq \frac{F_2}{G_2}$$

$$= N_\Delta \frac{e^{\lambda S_{ave}}}{\lambda} / e^{\lambda\,(T - 0.5\,\sigma)} \qquad \text{(from (7) and (13))}$$

$$= N_\Delta \frac{e^{\lambda\,(S_{ave} - T + 0.5\,\sigma)}}{\lambda} \tag{14}$$

Substituting N_Δ from E4(4) yields

$$\Lambda_{S2} = \frac{e^{-\frac{1}{2}\left(\frac{S_o - S_j}{\sqrt{2}\,\sigma}\right)^2}}{\sqrt{2\pi}\,\sqrt{2}\,\sigma\,\lambda}\, e^{+\lambda\,(S_{ave} - T + .5\,\sigma)} \tag{15}$$

E6 — Λ_{S1} DEVELOPMENT

$$\Lambda_{S1} = F_1/G_1 \tag{1}$$

F_1 Development

$$F_1 = \frac{\int_X N_o(1 - Q_j)\, P(X)\, dx}{\int_X N_o P(S_o)\, dx} \tag{2}$$

As shown in the development of F_2, the denominator is $P(S_o)$. Assuming that $P(X)$ is relatively flat, then $P(X)/P(S_o) = 1$ so that, (refer to Figure E3(a))

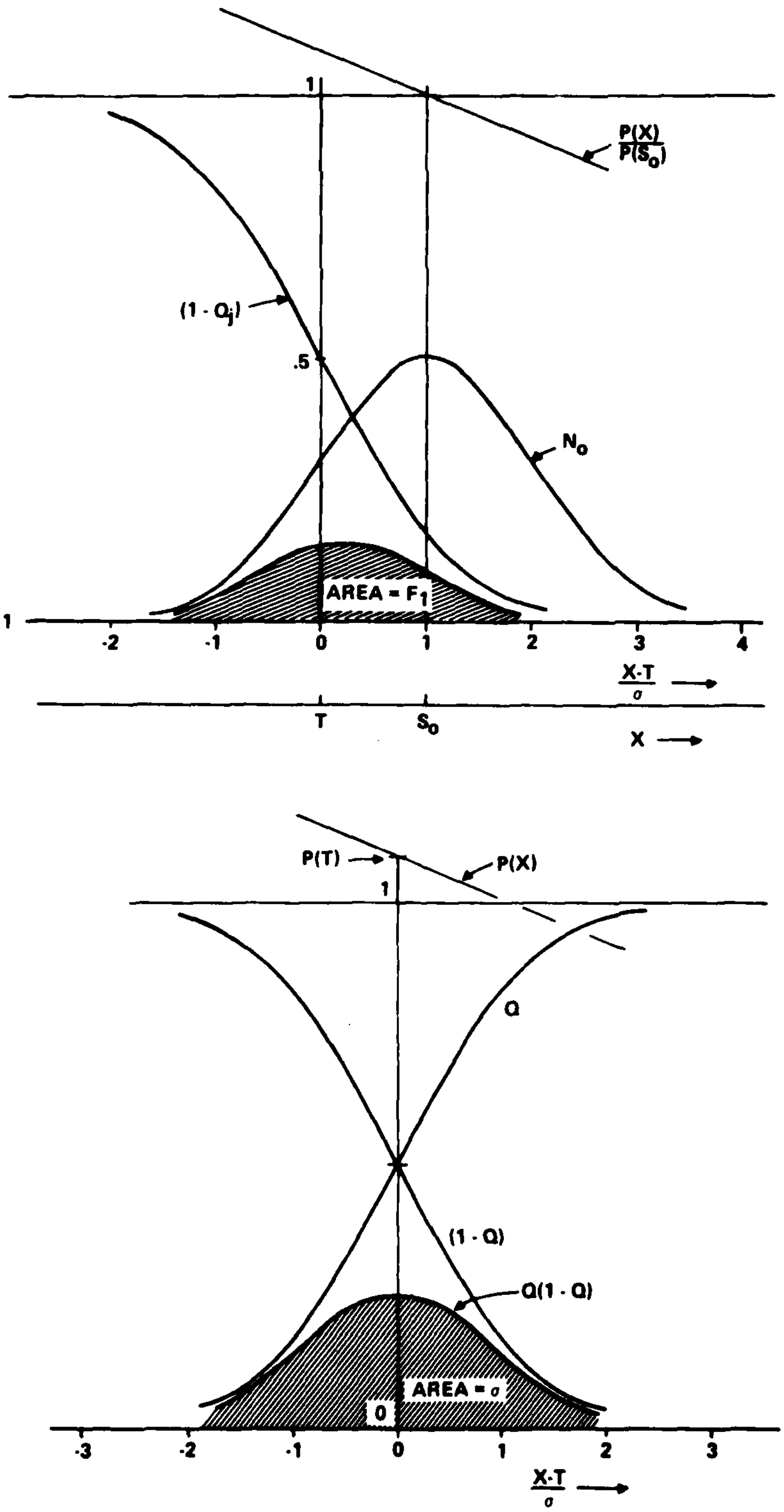
$$\begin{aligned} F_1 &= \int_X N_o\,(1 - Q_j)\, dx \\ &= 1 - \int_X N_o Q_j\, dx \end{aligned} \tag{3}$$

Note that Q_j is the integral of a normal distribution (N_j) integrated from the threshold (T) to ∞ (probability of detection given X). Then, substituting this integral and reversing the order of integration yields

$$F_1 = 1 - \int_T^{\infty} [\int_{-\infty}^{\infty} N_o N_j\, dx\,]\, dS_j \tag{4}$$

As shown in E4 the inner integral [ ] is a normal distribution of the amplitude difference $(S_j - S_o)$ so that

$$F_1 = 1 - \int_T^{\infty} N\left(\frac{S_o - S_j}{\sqrt{2}\,\sigma}\right) dS_j \tag{5.1}$$



(a) — $\underline{F_1}$

(b) — $\underline{G_1}$ (num)

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Figure E3 — F_1 and G_1

The integral is a Q function so that

$$F_1 = 1 - Q\left(\frac{T - S_o}{\sqrt{2}\,\sigma}\right) \quad (5.2)$$

Then, using the Q approximation of E3 yields

$$F_1 = 0.5\, e^{-0.8\left(\frac{S_o - T}{\sqrt{2}\,\sigma}\right)} \quad (6)$$

<u>G_1 Development</u>

$$G_1 = \frac{P(D\bar{D})}{P(D)} \quad (7)$$

Using the detection terms of E3 yields

$$G_1 = \frac{\int_X Q(1\text{-}Q)\, P(X)\, dx}{\int_X QP(X)\, dx}$$

$$= \frac{\int_o \left(1 - .5\, e^{\frac{-.8\, x'}{\sigma}}\right) \left(.5\, e^{\frac{-.8\, x'}{\sigma}}\right) P(X)\, dx}{\int_o (1 - .5\, e^{-.8\, x'})\; P(X)\, dx} \quad (8)$$

where

$$x' = (X - T)$$

Assume that P(X) is essentially linear in X and note that Q (1 – Q) is symmetrical about X = T (refer to <u>Figure E3(b)</u>). Then P(X) can be replaced by P(T) and integration yields the numerator term

$$G_1\,(\text{num}) = 2\, P(T) \left(\frac{-.5\, e^{-.8\, x/\sigma}}{.8\, \sigma} + \frac{.25\, e^{-1.6\, x/\sigma}}{1.6\, \sigma}\right) \Bigg]_{x = o}^{\infty}$$

$$= \frac{.75}{.8}\, \sigma\, P(T) \quad (10)$$

$$\cong \sigma\, P(T)$$

Consider the denominator term of (8). As developed for G_2 using the broad P(X), Q step function approximation and $P(X) = \lambda e^{-\lambda X}$, then,

$$G_1 \text{ (den)} = \int_T^\infty P(X)\,dx = e^{-\lambda T} = P(T)/\lambda \tag{11}$$

Finally, since $G_1 = G_1$ (num)/G_1 (den), then

$$G_1 = \lambda\sigma \tag{12}$$

E7 -- A PRIORI PROBABILITIES — λ

The problem here is to determine the a priori probability ratios ($\lambda_{m1}, \lambda_{m2}$), m = 0,1,2. These λ are a function of the probabilities $P(D_oD_j\ C|A)$ and $P(D_oD_j\ C|\overline{A})$.

Consider first the association condition (A) where a target and hence correlated lines can exist in the cells. The approach is to develop a probability tree enumerating all target and extraneous line conditions and the probabilities of each.

Define $\underline{T}$, $\overline{T}$ as the existance (detection) of a detectable, not detectable target line respectively. Similarly, for extraneous lines ($\underline{E}\ \overline{E}$).

If E_o is dominant over T_o (extraneous amplitude exceeds target amplitude), then T_o is treated as a non existing line ($\overline{T}_o$). Probability of this occurrence is 0.5 assuming T and E have the same amplitude distributions.

The probability tree enumerating all state/substates and their transition probabilities is shown in Figure E4.

Definitions

n_T = expected number of detectable target lines

n_c = number of cells

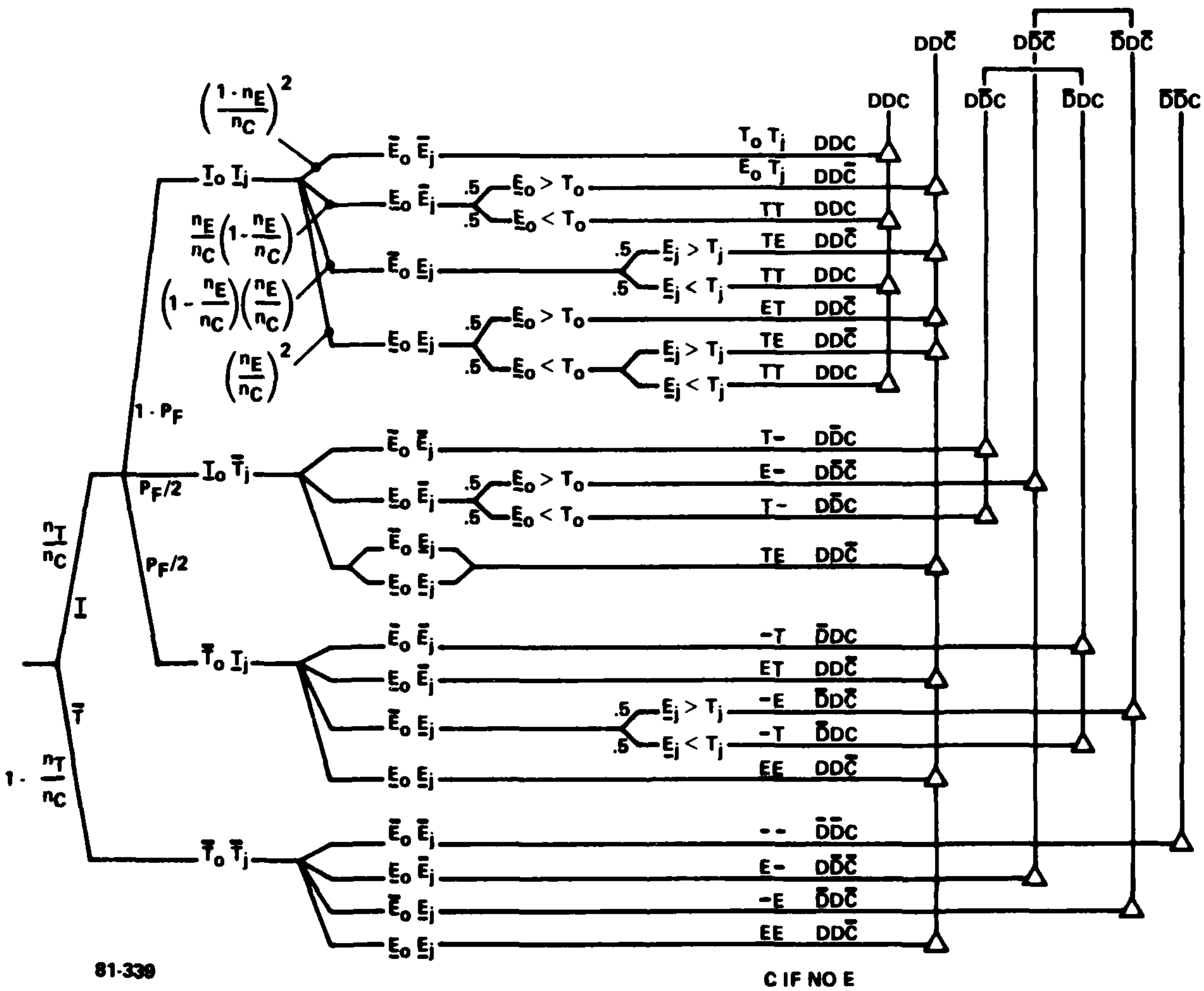

Figure E4 — Probability Tree for Association Condition

n_E = expected number of extraneous lines

P_F = probability of fade

At the initial node, as indicated, n_T/n_c is the probability of a target, (correlated line) being detected in either segment. Next, $(1 - P_F)$ is the probability of no fade given a line and hence detected in both segments ($\underline{TT}$). Given this condition, the probability of being in each of the four E states is indicated. Note the probability of no extraneous lines ($\overline{E}_o\overline{E}_j$) is $\left(1 - \frac{n_E}{n_c}\right)^2$. For the other three E states (an extraneous detection in one or both), an additional branch is shown. If the detectable extraneous line exceeds threshold ($E > T$), it is assumed that it will mask the correlated target line thereby resulting in the detections being uncorrelated ($DD\overline{C}$). As indicated, the probability of detection given a detectable E line is .5. The other branches and probabilities were similarly developed.

The final states are then grouped as indicated. The first group, for example, applies to a correlated detection in both cells (DDC).

The probability of occurrence of each of these final (detection/correlation) states is obtained by adding the probability of each path indicated, where the path probability is the product of all the branch probabilities back to the start.

The resulting probabilities are given in Table E-1. Recall that all of the above applies to the association condition, i.e., the existance of a correlated target in the two segments.

Nonassociation Condition

The comparable probabilities under the nonassociation condition ($\overline{A}$) are developed by a straightforward modification of the previous results for the A condition.

In each segment the expected number of target lines is given by

$$n_{To} = n_T (1 - P_{F/2})$$

TABLE E-1 — DDC A PRIORI PROBABILITY EXPRESSIONS FOR ASSOCIATED CONDITION

$$P(\underline{D}\underline{D}\underline{C}|\underline{A}) = \frac{n_T}{n_c}(1 - P_F)\left[\left(1 - \frac{n_E}{n_c}\right)^2 + 2(.5)\frac{n_E}{n_c}\left(1 - \frac{n_E}{n_c}\right) + (.5)(.5)\left(\frac{n_E}{n_c}\right)^2\right]$$

$$= \frac{n_T}{n_c}(1 - P_F)\left(1 - .5\frac{n_E}{n_c}\right)^2$$

$$\cong \frac{n_T}{n_c}(1 - P_F)$$

$$P(\underline{D}\underline{D}\overline{C}|\underline{A}) = \frac{n_E}{n_c}\left\{\frac{n_E}{n_c}\left[1 - \frac{n_T}{n_c}(1.25 - .25\, P_F)\right] + \frac{n_T}{n_c}\right\}$$

$$\cong \frac{n_E}{n_c^2}(n_E + n_T)$$

$$P(\underline{D}\overline{D}\underline{C}|\underline{A}) = \frac{n_T}{n_c}\,\frac{P_F}{2}\left[1 - 1.5\frac{n_E}{n_c} + .5\left(\frac{n_E}{n_c}\right)^2\right]$$

$$\cong \frac{n_T}{n_c}\,\frac{P_F}{2}$$

$$P(\underline{D}\overline{D}\overline{C}|\underline{A}) = \left[1 - \frac{n_T}{n_c}\left(1 - \frac{P_F}{4}\right)\right]\frac{n_E}{n_c}\left(1 - \frac{n_E}{n_c}\right)$$

$$\cong \frac{n_E}{n_c}$$

$$P(\overline{D}\underline{D}\underline{C}|\underline{A}) = P(\underline{D}\overline{D}\underline{C}|\underline{A})$$

$$P(\overline{D}\underline{D}\overline{C}|\underline{A}) = P(\underline{D}\overline{D}\overline{C}|\underline{A})$$

$$P(\overline{DD}|\underline{A}) = \left(1 - \frac{n_T}{n_c}\right)\left(1 - \frac{n_E}{n_c}\right)^2$$

Add this to n_E to obtain the new n_E as given by

$$n_{Eo} = n_E + n_T (1 - P_{F/2}) \triangleq n_E'$$

and set $n_T = 0$ in the previous association analysis. The resulting probabilities are given below in Table E-2.

TABLE E-2 — DDC EXPRESSIONS FOR NONASSOCIATED CONDITION

$$P(\underline{DDC}|\overline{A}) = 0$$

$$P(\underline{DD}\overline{C}|\overline{A}) = \left(\frac{n_E'}{n_c}\right)^2$$

$$P(\underline{D}\overline{D}\underline{C}|\overline{A}) = 0$$

$$P(\underline{D}\overline{D}\overline{C}|\overline{A}) = \frac{n_E'}{n_c}\left(1 - \frac{n_T'}{n_c}\right)$$

$$P(\overline{D}\overline{D}\overline{C}|\overline{A}) = \left(1 - \frac{n_E'}{n_c}\right)^2$$

As previously defined, the λ's are defined as follows and the above probabilities substituted to obtain the results of Table E-3.

TABLE E-3 — λ

$$\lambda_{21} = \frac{P(\underline{DDC}|A)}{P(\underline{DD}\overline{C}|\overline{A})} \cong \frac{n_T (1 - P_F)\, n_c}{(n_E')^2}$$

$$\lambda_{22} = \frac{P(\underline{DD}\overline{C}|A)}{P(\underline{DD}\overline{C}|\overline{A})} \cong \frac{n_E (n_E + n_T)}{(n_E')^2}$$

$$\lambda_{11} = \frac{P(\underline{D}\overline{D}\underline{C}|A)}{P(\underline{D}\overline{D}\overline{C}|\overline{A})} = \frac{n_T P_F}{2 n_E'}$$

TABLE E-3 — λ (CONT'D)

$$\lambda_{12} = \frac{P(\underline{D}\overline{D}\overline{C}|A)}{P(\underline{D}\overline{D}\overline{C}|\overline{A})} \approx \frac{n_E}{n_E'}$$

$$\lambda_o = \frac{P(\overline{D}\overline{D}|A)}{P(\overline{D}\overline{D}|\overline{A})} \cong 1 + \frac{n_T(1 - P_F)}{n_c}$$

E8—APPROXIMATION TO $\prod^{n_o} \lambda_o$

The problem is to obtain an approximation for $F = \prod^{n_o} \lambda_o$. As given by the final expression of E7

$$\lambda_o = 1 + x \tag{1}$$

where

$$x = \frac{n_T(1 - P_F)}{n_c} \tag{2}$$

thus

$$\begin{aligned} F &= \prod^{n_o} \lambda_o = {\lambda_o}^{n_o} \\ &= (1 + x)^{n_o} \end{aligned} \tag{3}$$

where n_o is the number of cells with no detections as given by

$$n_o = n_c - (n_1 - n_2) \tag{4}$$

Taking natural logs (ln) yields

$$\ln F = n_o \ln (1 + x) \tag{5}$$

Expanding ln (1 + x) in a Taylor series noting that x is small (of the order of 0.01) yields

$$\ln F = n_o [x - x^2/2 + \ldots] \tag{6}$$

Neglecting x^2 and succeeding terms and substituting (2) and (4) yields

$$
\begin{aligned}
\ln F &= [n_c - (n_1 + n_2)] \frac{n_T (1 - P_F)}{n_c} \\
&\cong n_T (1 - P_F)
\end{aligned}
\tag{7}
$$

Taking the antilog yields

$$F = e^{n_T (1 - P_F)} \tag{8}$$

E9—EVALUATION OF $\Lambda_{S2_{tot}}$

The total frequency likelihood ratio for cells which match (detection in both segments) is given by

$$\Lambda_{S2_{tot}} = \Lambda_{S2} \lambda_{21} + \lambda_{22} \tag{1}$$

Λ_{S2} is given by (15) of E5 and $\lambda_{21}, \lambda_{22}$ by Table E3 of E7. Substitution yields

$$\Lambda_{S2_{tot}} = \Lambda_{S2} \left[\frac{(1 - P_F)\, n_T n_c}{[n_E + (1 - P_F/2)\, n_T]^2} \right] + \frac{n_E (n_E + n_T)}{[n_E + (1 - P_F/2)\, n_T]^2} \tag{2}$$

For insight assume that P_F is small and define r_c and r_E as the normalized values with respect to n_T

$$
\begin{aligned}
r_c &= n_c/n_T \\
r_E &= n_E/n_T
\end{aligned}
$$

then,

$$\Lambda_{S2_{tot}} = \Lambda_{S2} \frac{r_c}{(1 + r_E)^2} + \frac{r_E}{1 + r_E} \tag{3}$$

Clipper Processor

For the clipper processor, $\Lambda_{S2} = 1$. Let $r_c = 200$. Note that λ_{22} is significant only for large values of r_E and then it's value is unity. Thus,

$$\Lambda^c_{S2_{tot}} = \frac{r_c}{(1 + r_E)^2} + 1 \tag{4}$$

$$= \frac{n_c n_T}{(n_T + n_E)^2} + 1$$

Amplitude Processor

If amplitude is available, then as given by (15) of E5, the match likelihood ratio is

$$\Lambda_{S2} = \frac{e^{-\frac{1}{2}\Delta S'^2}}{\sqrt{2\pi}\sqrt{2}\,P_F} \quad e^{P_F(\delta S' + .5)} \tag{5}$$

where,

$$\Delta S' = \frac{S_o - S_j}{\sqrt{2}\,\sigma_F} = \text{normalized amplitude difference}$$

$$\delta S' = \frac{\frac{S_o + S_j}{2} - T}{\sigma_F} = \text{normalized average amplitude relative to threshold}$$

Note the identity substitution $P_F = \lambda\sigma_F$ has been made.

Then,

$$\Lambda_{S2_{tot}} \cong \left[\Lambda_{S2}\,\frac{r_c}{(1 + r_E)^2} + 1\right] \tag{6}$$

Figure E5 is a plot of Λ_{S2} vs P_F for various ΔS, δS. Note that for practical fade probabilities say 0.1 and 0.3, Λ is fairly sensitive to P_F.

Figure E6 shows sensitivity to ΔS and various δS with P_F set at 0.2.

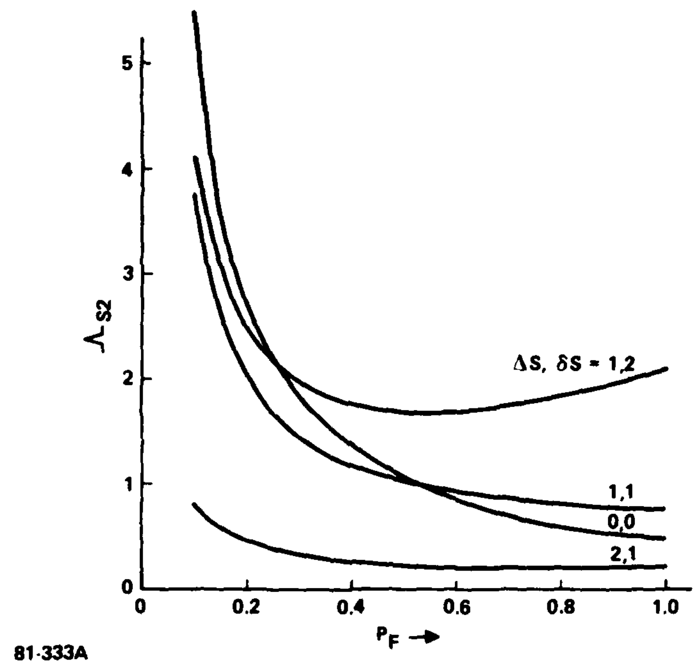

Figure E5 — Λ_{S2} vs P_F

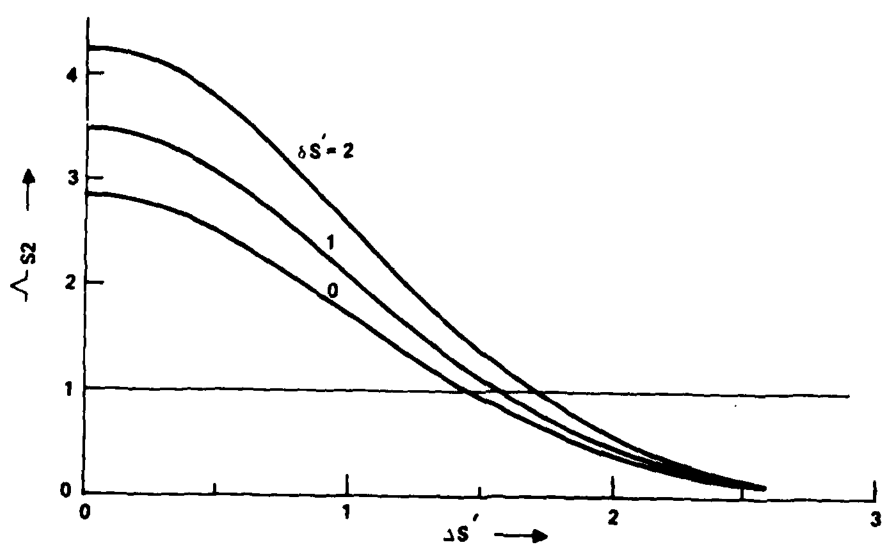

Figure E6 — Λ_{S2} vs ΔS

APPENDIX F — LIKELIHOOD RATIO FOR BINARY ASSOCIATION — HYPOTHESIS TEST WITH LINE DETECTIONS AND MATCHES AS MEASUREMENTS

— An Initial, Simplified, Frequency Likelihood Approach —

INTRODUCTION

In this appendix there is presented a combinatorially oriented derivation of the likelihood ratio for testing the association between track segments based solely on measurements conveying the detection or nondetection of matching tonals.

The treatment differs most importantly from that found in Appendices D and E in that,

i. The a priori information is given a different parametric description,

ii. extraneous lines are here not permitted (false-detection probability from noise alone is assumed zero), and

iii. the amplitudes of the detections do not here constitute part of the measurements.

As a consequence of iii, the development here should be compared to the "clipped case" development of Appendices D and E. In making that comparison, it is seen that in both cases the log likelihood ratio is a linear function of the same two measurements; the number of matched lines between segments, and the total number of lines, matched and unmatched, in the two segments.

MODEL

A source is characterized by n_s tonals. The receiver has n_c FFT bins; at times t_y and (later) t_z, n_y and (later) n_z tones are detected, and n_2 of the n_y and n_z "match" in frequency. Hypotheses are:

H_1 — Same source at t_y and t_z

H_0 — Different sources at t_y and t_z.

Note in the previous analysis, t_o and t_j were used rather than t_z and t_y, respectively.

Likelihood ratio is

$$\Lambda(n_2,n_y,n_z) = \frac{P_{H1}(n_2,n_y,n_z)}{P_{Ho}(n_2,n_y,n_z)} \tag{1}$$

ASSUMPTIONS

1. n_s has Poisson distribution with mean m;
2. Source tonals are distributed at random across FFT band;
3. p = probability of detection, given that a source tone is present, is same for every tone;
4. Doppler shifts ignored, so that "match" means detection in same FFT bin at both t_y and t_z;
5. Negligible false-detection probability from noise alone; and
6. $n_c >> n_y$, n_z, or n_2.

ANALYSIS

Under H_1, n_y and n_z are the number of successes in two independent experiments of n_s Bernoulli trials each, with single-trial probability of success given by

$$p \quad = \text{Prob}\left(\begin{matrix}\text{line}\\ \text{detected}\end{matrix}\right) \tag{2}$$

for each of the n_s cells.

Under H_o, n_y is the number of successes in n_c trials with single-trial probability of success given by

$$P_y \quad = p\,\frac{n_{sy}}{n_c} \tag{3}$$

and similarly for n_z, with

$$P_z \quad = p\,\frac{n_{sz}}{n_c} \tag{4}$$

where n_{sy} and n_{sz} each has a Poisson distribution with mean m.

Note that under H_1, n_2 is distributed as the number of red balls obtained when n_z balls are drawn without replacement from an urn containing n_s balls of which n_y are red, so that

$$P(n_2|n_y, n_z, n_s) = \frac{\binom{n_y}{n_2}\binom{n_s - n_y}{n_z - n_2}}{\binom{n_s}{n_z}} \tag{5}$$

while under H_o, n_2 is distributed as in (5) but with n_s replaced by n_c.

UNDER H_1

$$P_{H_1}(n_2, n_y, n_z) = \sum_{n_s=0}^{\infty} P(n_2, n_y, n_z, n_s)$$

$$= \sum_{n_s=o}^{\infty} \underbrace{P(n_2|n_y, n_z\, n_s)} \qquad \underbrace{P(n_y|n_s)} \qquad \underbrace{P(n_z|n_s)} \qquad \underbrace{P(n_s)}$$

$$= \sum_{n_s=0}^{\infty} \frac{\binom{n_y}{n_2}\binom{n_s - n_y}{n_z - n_2}}{\binom{n_s}{n_z}} \binom{n_s}{n_y} p^{n_y} q^{n_s n_y} \binom{n_s}{n_z} p^{n_z} q^{n_s - n_z} \frac{m^{n_s} e^{-m}}{n_s!} \tag{6}$$

where

$$q \overset{\Delta}{=} 1 - p.$$

Expanding (6) with

$$\binom{k}{r} = \frac{k!}{r!\,(k-r)!} \tag{7}$$

and using

$$\sum_{n_s=o}^{\infty} \frac{(mq^2)^{n_s}}{(n_s - n_y + n_z - n_2)!} = (mq^2)^{(n_y + n_z - n_2)} e^{mq^2} \tag{8}$$

yields

$$P_{H_1}(n_2, n_y, n_z) = \frac{\left(\frac{p}{q}\right)^{n_y + n_z}}{n_2!\,(n_y - n_2)!\,(n_z - n_2)!} \quad (mq^2)^{(n_y + n_z - n_2)} \cdot e^{m(q^2 - 1)} \tag{9}$$

UNDER H_o

$$P_{H_o}(n_2,n_y,n_z) = \sum_{n_y,n_z=0}^{\infty}\sum P(n_2,n_y,n_z,n_{sy},n_{sz}) \tag{10}$$

$$= \sum_{n_{sy},n_{sz}=0}^{\infty}\sum P(n_2|n_y,n_z,H_o) \quad P(n_y|n_{sy}) \quad P(n_z|n_{sz}) \quad P(n_{sy}) \quad P(n_{sz}) \tag{11}$$

$$\cong \sum_{n_{sy},n_{sz}=0}^{\infty}\sum \frac{\binom{n_y}{n_2}\binom{n_c-n_y}{n_z-n_2}}{\binom{n_c}{n_z}} \frac{(pn_{sy})^{n_y} e^{-pn_{sy}}}{n_y!} \frac{(pn_{sz})^{n_z} e^{-pn_{sz}}}{n_z!} \frac{m^{n_{sy}} e^{-m}}{n_{sy}!} \frac{m^{n_{sz}} e^{-m}}{n_{sz}!}$$

where we have used the Poisson approximation to the Bernoulli distribution for $P(n_y|n_{sy})$ and $P(n_z|n_{sz})$, e.g.,

$$P(n_y|n_{sy}) = \binom{n_c}{n_y}\left(p\frac{n_{sy}}{n_c}\right)^{n_y}\left(1-p\frac{n_{sy}}{n_c}\right)^{n_c-n_y}$$
$$\approx \frac{(pn_{sy})^{n_y} e^{-pn_{sy}}}{n_y!} \tag{12}$$

Evaluating (11), we obtain

$$P_{H_o}(n_2,n_y,n_z) = \frac{(n_c-n_y)!\,(n_c-n_z)!\,p^{n_y+n_z} e^{2m(e^{-p}-1)}}{n_2!(n_y-n_2)!\,(n_z-n_2)!\,(n_c-[n_y+n_z-n_2])!\,n_c!} \nu_{n_y}(\mu)\,\nu_{n_z}(\mu) \tag{13}$$

where

$$\mu \triangleq me^{-p} \tag{14}$$

and $\nu_r(\mu)$ is the r^{th} absolute moment of a Poisson variate with mean μ

$$\nu_r(\mu) = \sum_{k=0}^{\infty} \frac{k^r \mu^k e^{-\mu}}{k!} \tag{15}$$

so that(*)

r	$\nu_r(\mu)$
0	1
1	μ
2	$\mu(\mu + 1)$
3	$\mu(\mu^2 + 3\mu + 1)$
4	$\mu(\mu^3 + 6\mu^2 + 7\mu + 1)$
...	...

Combining (9,13-15) gives

$$\Lambda\,(n_2,n_y,n_z) = \Gamma \cdot \frac{(mq)^{n_y + n_z}}{(mq^2)^{n_2}} \cdot \frac{e^{m[2(1-e^{-p}) - p(2-p)]}}{\nu_{n_y}(me^{-p})\,\nu_{n_z}(me^{-p})} \tag{16}$$

where

$$\Gamma = \frac{n_c!\,(n_c - [n_y + n_z - n_2])!}{(n_c - n_y)!\quad(n_c - n_z)!} \tag{17}$$

Using Stirling's approximation and

$$(1 + 1/n)^n \approx e \text{ for } n >> 1$$

we have, for $n_c >> n_y$, n_z or n_2,

$$\Gamma \approx \left(\frac{n_c}{e^2}\right)^{n_2}, \quad e = 2.71828... \tag{18}$$

and (16) becomes

$$\Lambda\,(n_2,n_y,n_z) \approx \left(\frac{n_c}{mq^2e^2}\right)^{n2} \cdot \frac{(mq)^{n_y + n_z}}{\nu_{n_y}(me^{-p})\,\nu_{n_z}(me^{-p})} \cdot e^{m\,[2\,(1-e^{-p}) - p\,(2-p)]} \tag{19}$$

(*) Burington & May, "Handbook of Prob. & Stat." Handbook Pub., Sandusky, Oh 1953; pg. 77.

Note that (19) is dominated by the value of n_2, and is symmetric in n_y and n_z. Figure F1 sketches for various combinations of n_2, n_y, and n_z the variation of log Λ as a function of p with n_c = 400 and m = 5.

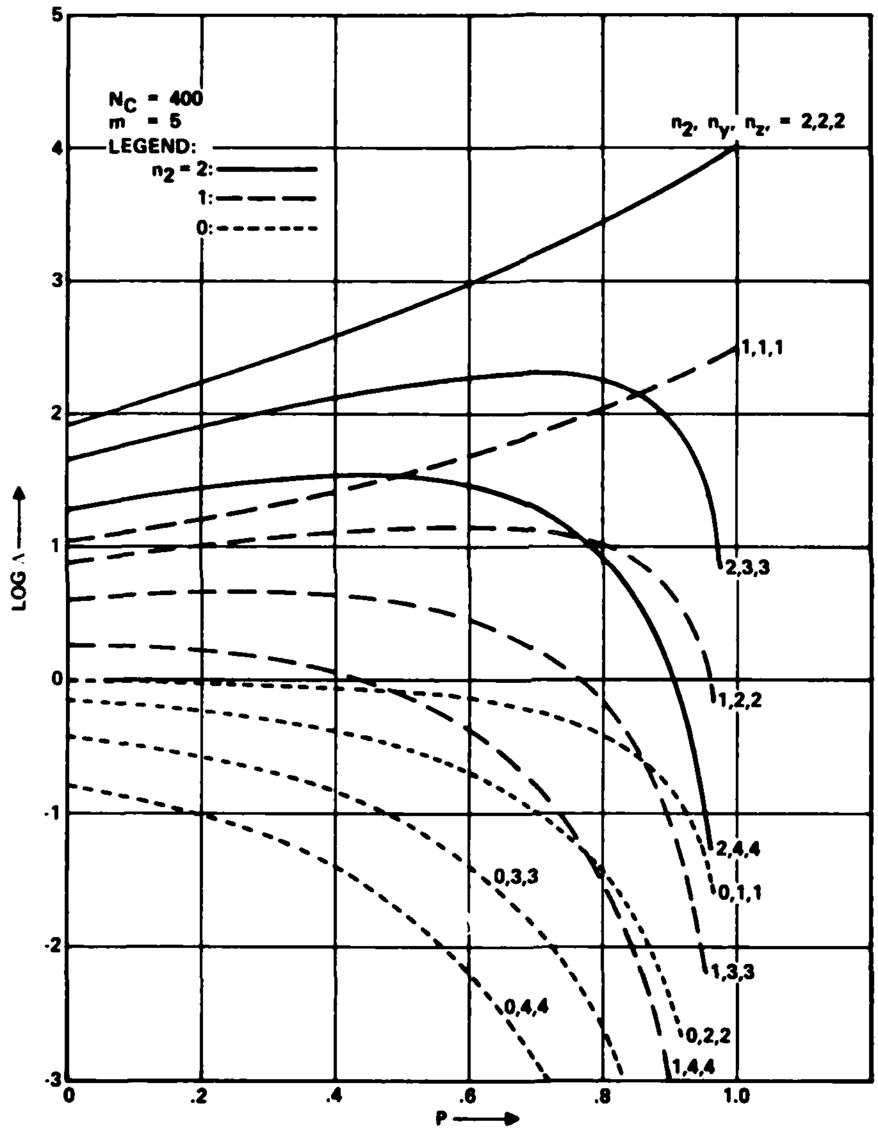

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Figure F1 — Log Λ

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